Free Form Surface Design with A-Patches*

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Abstract

We present a sufficient criterion for the Bernstein-Bezier (BB) form of a trivariate polynomial within a tetrahedron, such that the real zero contour of the polynomial defines a smooth and single sheeted algebraic surface patch. We call this an A-patch. We present algorithms to build a mesh of cubic A-patches to interpolate a given set of scattered point data in three dimensions, respecting the topology of any surface triangulation T of the given point set. In these algorithms we first specify "normals" on the data points, then build a simplicial hull consisting of tetrahedra surrounding the surface triangulation T and finally construct cubic A-patches within each tetrahedron. The resulting surface constructed is C^1 (tangent plane) continuous and single sheeted in each of the tetrahedra. We also show how to adjust the free parameters of the A-patches to achieve both local and global shape control.

Keywords : Algebraic Surface Patches, Interpolation, Approximation, C^1 Continuity

1 Introduction

The importance of implicit surface representation in modeling geometric objects or reconstructing the image to scattered data has been described in various papers (see for e.g. [2, 7, 9, 11, 16]). The main shortcoming held against the popular use of implicit surfaces is that the representation being multivalued may cause the real zero contour surface to have multiple sheets, self-intersections and several other undesirable singularities.

In section 3 of this paper, we present a sufficient criterion for the Bernstein-Bezier (BB) form of a trivariate polynomial within a tetrahedron such that the real zero-contour of the polynomial is smooth (non-singular) and a single sheeted algebraic surface. We call this an A-patch. In section 4, we describe how to build a simplicial hull consisting of tetrahedra surrounding a surface triangulation T of the set of scattered data points in 3D. We then show in section 5 how a mesh of cubic A-patches can be used to construct a C^1 interpolatory surface, respecting the topology of the surface triangulation T. In section 6, we show how to adjust the free parameters of the A-patches to achieve both local and global shape control. This C^1 cubic A-patch fitting algorithm is quite appropriate for free form design. In analogy to the final smoothing of an artist's rough sketches, complicated smooth models can be directly formed by first creating a rough polyhedral model of the desired object and then using the fitting algorithms to produce a C^1 smooth solid with extra local and global parameters for fine shape control. Proofs of all theorems and lemmas are given in the full version of this paper [3].

Related Prior Work:

The work of characterizing the BB form of polynomials within a tetrahedron such that the zero contour of the polynomial is a single sheeted surface within the tetrahedron, has been attempted in the past. In [16], Sederberg showed that if the coefficients of the BB form of the trivariate polynomial on the lines that parallel one edge, say L, of the tetrahedron, all increase (or decrease) monotonically in the same direction, then any line parallel to L will intersect the zero contour algebraic surface patch at most once. In [9], Guo treats the same problem by enforcing monotonicity conditions on a cubic polynomial along the direction from one vertex to a point of the opposite face of the vertex. From this he derives a condition $a_{\lambda-e_1+e_4} - a_\lambda \geq 0$ for all $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)^T$ with $\lambda_1 \geq 1$, where a_λ are the coefficients of the cubic in BB form and e_i is the i-th unit vector. This condition

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is difficult to satisfy in general, and even if this condition is satisfied, one still cannot avoid singularities on the zero contour. Our condition of a smooth, single sheeted zero contour in Theorem 3.2 of §3 generalizes Sederberg's condition and provides us with an efficient way of generating A-patches.

The second problem we consider is how to join a collection of A-patches to form a C^1 smooth surface interpolating scattered data points and respecting the topology of a given surface triangulation T of the points. For this problem, prior approaches have been given by [6] using quadric patches, [7, 9, 10] using cubic patches and [4] using quintic for convex triangulations and degree seven patches for arbitrary surface triangulations T. All these papers provide heuristics to overcome the multiple sheeted and singularity problems of implicit patches. In this paper our cubic A-patches are guaranteed to be nonsingular and single sheeted within each tetrahedron.

While the details of the methods of [7] and [10] differ somewhat, they both use the scheme of [6] of building a surrounding simplicial hull (consisting of a series of tetrahedra) of the given triangulation T. Such a simplicial hull is nontrivial to construct for triangulations and neither of the papers [6, 7, 9, 10] enumerate the different exceptional cases (possible even for convex triangulations) nor provide solutions to overcoming them. We too use the simplicial hull approach in this paper but enumerate the exceptional situations and provide some heuristic strategies for rectifying them.

In [10], Guo uses a Clough-Tocher split[5] and subdivides each face tetrahedron of the simplicial hull, hence utilizing three patches per face of T. In this paper, we consider the computed "normals" at the given data points, and distinguish between "convex" and "non-convex" faces and edges of the triangulation. These concepts are formally defined in section 4. We use a single cubic A-patch per face of T except for the following two special cases. For a non-convex face, if additionally the three inner products of the face normal and its three adjacent face normals have different signs, then in this case one needs to subdivide the face using a single Clough-Tocher split, yielding C^1 continuity with the help of three cubic A-patches for that face. Furthermore for coplanar adjacent faces of T, we show that the C^1 conditions cannot be met using a single cubic A-patch for each face. Hence for this case we again use Clough-Tocher splits for the pair of coplanar faces yielding C^1 continuity with the help of three cubic A-patches per face. See also the examples and figures in section 7 where the savings in patches becomes evident.

Related papers which approximate scattered data using implicit algebraic patches are [1, 11, 12] and a classification of data fitting using parametric surface patches is given in [14].

2 Notation and Preliminary Details

Problem Given a list of data points $P = \{p_1, \dots p_k\} \in \mathbb{R}^3$ and a surface triangulation T of these points, construct a mesh of low degree algebraic surfaces such that the composite surface is single sheeted C^1 continuous and has the same topology as T.

Convex Hull, Affine Hull: Let $\{p_1, \dots, p_j\} \in \mathbb{R}^3$ with $j \le 4$. Then the *convex hull* of these points is defined by $[p_1 p_2 \dots p_j] = \{p \in \mathbb{R}^3 : p = \sum_{i=1}^{j} \alpha_i p_i, \alpha_i \ge 0, \sum_{i=1}^{j} \alpha_i = 1\}$ and the *affine hull* is defined by $\langle p_1 p_2 \dots p_j \rangle = \{p \in \mathbb{R}^3 : p = \sum_{i=1}^{j} \alpha_i p_i, \sum_{i=1}^{j} \alpha_i = 1\}$. The interior of the convex hull $[p_1 p_2 \dots p_j]$ is denoted by $(p_1 p_2 \dots p_j) = \{p \in \mathbb{R}^3 : p = \sum_{i=1}^{j} \alpha_i p_i, \alpha_i > 0, \sum_{i=1}^{j} \alpha_i = 1\}$.

Bernstein-Bezier (BB) Form: Let p_1, p_2, p_3, $p_4 \in \mathbb{R}^3$ be affine independent. Then the tetrahedron with vertices p_1, p_2, p_3, and p_4, is $V = [p_1 p_2 p_3 p_4]$. For any $p = \sum_{i=1}^{4} \alpha_i p_i \in V, \alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T$ is the barycentric coordinate of p. Let $p = (x, y, z)^T$, $p_i = (x_i, y_i, z_i)^T$. Then the barycentric coordinates relate to the Cartesian coordinates via the following relation

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} \tag{1}$$

Any polynomial $f(p)$ of degree n can be expressed as Bernstein-Bezier(BB) form over V as $f(p) = \sum_{|\lambda|=n} b_\lambda \, B_\lambda^n(\alpha)$, $\lambda \in \mathcal{Z}_+^4$, where $B_\lambda^n(\alpha) = \frac{n!}{\lambda_1!\lambda_2!\lambda_3!\lambda_4!} \alpha_1^{\lambda_1}\alpha_2^{\lambda_2}\alpha_3^{\lambda_3}\alpha_4^{\lambda_4}$ is Bernstein polynomial, $|\lambda| = \sum_{i=1}^{4} \lambda_i$ with $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)^T$, $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T = \sum_{i=1}^{4} \alpha_i e_i$ is barycentric coordinate of p, $b_\lambda = b_{\lambda_1\lambda_2\lambda_3\lambda_4}$(as a subscript, we simply write λ as $\lambda_1\lambda_2\lambda_3\lambda_4$) are called control points, and $\mathcal{Z}_+^4$ stands for the set of all four dimensional vectors with nonnegative integer components. The following basic facts about the BB form will be used in this paper. The first is derived from the directional derivative formulas(see [8]).

Lemma 2.1. *If $f(p) = \sum_{|\lambda|=n} b_\lambda B_\lambda^n(\alpha)$, then*

$$b_{(n-1)e_i+e_j} = b_{ne_i} + \frac{1}{n}(p_j - p_i)^T \nabla f(p_i), \quad j = 1 \dots 4, j \ne i \tag{2}$$

where $\nabla f(p) = [\frac{\partial f(p)}{\partial x} \; \frac{\partial f(p)}{\partial y} \; \frac{\partial f(p)}{\partial z}]^T$

Formula (2) will be used to determine the control points around a vertex from the given normal at that vertex.

Lemma 2.2 ([8]). *Let $f(p) = \sum_{|\lambda|=n} a_\lambda B^n_\lambda(\alpha)$ and $g(p) = \sum_{|\lambda|=n} b_\lambda B^n_\lambda(\alpha)$ be two polynomials defined on two tetrahedra $[p_1p_2p_3p_4]$ and $[p'_1p_2p_3p_4]$, respectively. Then*
(i) f and g are C^0 continuous at the common face $[p_2p_3p_4]$ if and only if

$$a_\lambda = b_\lambda, \quad for\ any\ \lambda = 0\lambda_2\lambda_3\lambda_4, \quad |\lambda| = n \qquad (3)$$

(ii) f and g are C^1 continuous at the common face $[p_2p_3p_4]$ if and only if (3) holds and

$$b_{1\lambda_2\lambda_3\lambda_4} = \beta_1 a_{1\lambda_2\lambda_3\lambda_4} + \beta_2 a_{0\lambda_2\lambda_3\lambda_4+0100} + \beta_3 a_{0\lambda_2\lambda_3\lambda_4+0010} + \beta_4 a_{0\lambda_2\lambda_3\lambda_4+0001} \qquad (4)$$

where $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)^T$ are defined by the relation $p'_1 = \beta_1 p_1 + \beta_2 p_2 + \beta_3 p_3 + \beta_4 p_4$, $|\beta| = 1$. Relation (4) will be called the coplanar condition.
Degree Elevation. The polynomial $f(p) = \sum_{|\lambda|=n} b_\lambda\, B^n_\lambda(\alpha)$ can be written as one of degree $n+1$ (see e.g. [8]): $f(p) = \sum_{|\lambda|=n+1} (Eb)_\lambda\, B^{n+1}_\lambda(\alpha)$, $\lambda \in \mathcal{Z}^4_+$, where $(Eb)_\lambda = \frac{1}{n+1}\sum_{i=1}^4 \lambda_i b_{\lambda-e_i}$.
Variation Diminishing Property ([8],p.54). For $y(t) = \sum_{i=0}^n b_i B^n_i(t)$, the number of intersections (counting multiplicities) with any line is no more than the intersections of any line with the polygon $\{\frac{i}{n}, b_i\}_{i=0}^n$ in [0, 1].

3 Sufficient Conditions of an A-Patch

Let $F(\alpha) = \sum_{|\lambda|=n} b_\lambda B^n_\lambda(\alpha)$ be a given polynomial of degree n on the simplex(tetrahedron) $S = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T \in I\!R^4 : \sum_{i=1}^4 \alpha_i = 1,\ \alpha_i \geq 0\}$. The surface patch within the simplex is defined by $S_F \subset S : F(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = 0$. The following two conditions on the trivariate BB-form will be used in this paper.
Smooth vertices condition. For each $i(1 \leq i \leq 4)$, there is at least one non-zero $b_{\lambda_1\lambda_2\lambda_3\lambda_4}$ for $\lambda_i \geq n-1$.
Smooth edges condition. For each pair $(i,j)(1 \leq i,j \leq 4, i \neq j)$, there is either at least one non-zero $b_{me_i+(n-m)e_j}$ for $m = 0, 1, \cdots, n$, or the polynomials $\sum_{m=0}^{n-1} b_{me_i+(n-1-m)e_j+e_k} B^{n-1}_m(t)$ and $\sum_{m=0}^{n-1} b_{me_i+(n-1-m)e_j+e_l} B^{n-1}_m(t)$ have no common zero in [0, 1], for distinct i, j, k, l.

If the surface S_F contains a vertex/edge, then it is easy to show by the formulas of directional derivatives(see [8], p. 312) that the surface is smooth there if the smooth vertex/edge conditions above are satisfied.
Definition 3.1. *Three-sided patch.*

Let the surface patch S_F be smooth on the boundary of the tetrahedron S. If any open line segment (e_j, α^*)

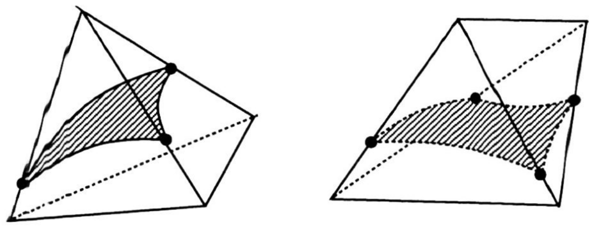


Figure 1: Three Sided and Four Sided Patches

with $\alpha^* \in S_j = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T : \alpha_j = 0, \alpha_i > 0, \sum_{i\neq j} \alpha_i = 1\}$ intersects S_F at most once(counting multiplicities), then we call S_F a *three-sided j-patch* (see Figure 1).
Definition 3.2. *Four-sided patch.*

Let the surface patch S_F be smooth on the boundary of the tetrahedron S. Let (i, j, k, ℓ) be a permutation of (1,2, 3,4). If any open line segment (α^*, β^*) with $\alpha^* \in (e_ie_j)$ and $\beta^* \in (e_ke_\ell)$ intersects S_F at most once(counting multiplicities), then we call S_F a *four-sided ij-kℓ-patch* (see Figure 1).

It is easy to see that if S_F is a four-sided ij-$k\ell$-patch, it is then also a ji-ℓk-patch, a ℓk-ji-patch, and so on. The full version of this paper [3] contains proofs of the lemmas and theorems below.
Lemma 3.1. *The three-sided j-patch and the four-sided ij-kℓ-patch are smooth (non-singular).*
Theorem 3.2. *Let $F(\alpha) = \sum_{|\lambda|=n} b_\lambda B^n_\lambda(\alpha)$ satisfy the smooth vertex and smooth edge conditions and $j(1 \leq j \leq 4)$ be a given integer. If there exists an integer $k(0 \leq k < n)$ such that*

$$b_{\lambda_1\lambda_2\lambda_3\lambda_4} \geq 0, \qquad \lambda_j = 0, 1, \ldots, k-1, \qquad (5)$$

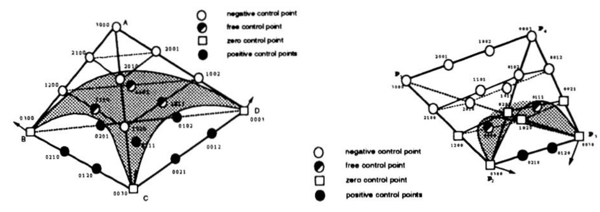
$$b_{\lambda_1\lambda_2\lambda_3\lambda_4} \leq 0, \qquad \lambda_j = k+1, \ldots, n \qquad (6)$$

and $\sum_{\substack{|\lambda|=n\\ \lambda_j=0}} b_\lambda > 0$ if $k > 0$, $\sum_{\substack{|\lambda|=n\\ \lambda_j=m}} b_\lambda < 0$ for at least one $m(k < m \leq n)$, then S_F is a three-sided j-patch.
Theorem 3.3. *Let $F(\alpha) = \sum_{|\lambda|=n} b_\lambda B^n_\lambda(\alpha)$ satisfy the smooth vertex and smooth edge conditions and (i, j, k, ℓ) be a permutation of (1, 2, 3, 4). If there exists an integer $k(0 \leq k < n)$ such that*

$$b_{\lambda_1\lambda_2\lambda_3\lambda_4} \geq 0; \qquad \lambda_i + \lambda_j = 0, 1, \ldots, k-1, \qquad (7)$$

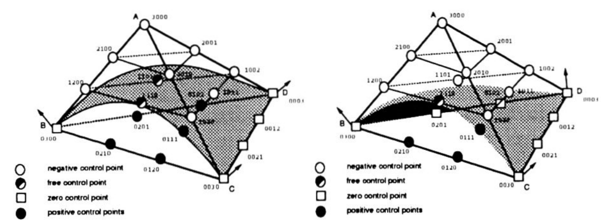
$$b_{\lambda_1\lambda_2\lambda_3\lambda_4} \leq 0; \qquad \lambda_i + \lambda_j = k+1, \ldots, n \qquad (8)$$

and $\sum_{\substack{|\lambda|=n\\ \lambda_i+\lambda_j=0}} b_\lambda > 0$ if $k > 0$, $\sum_{\substack{|\lambda|=n\\ \lambda_i+\lambda_j=m}} b_\lambda < 0$ for at least one $m(k < m \leq n)$, then S_F is four-sided ij-kℓ-patch.
Note. The conditions on the coefficients b_λ in Theorems 3.2 and 3.3 are sufficient but not necessary. For example



(a) (b)

Figure 2: (a) A Three Sided Patch Tangent at p_1, p_2, p_3 (b) A Degenerate Four Sided Patch Tangent to Face $[p_1p_2p_4]$ at p_2 and $[p_1p_3p_4]$ at p_3



(a) (b)

Figure 3: (a) A Three Sided Patch Interpolating the Edge $[p_2p_3]$ (b) A Three Sided Patch Interpolating Edges $[p_2p_3]$ and $[p_1p_3]$

if we want some $B_l < 0$, it is not necessary to let every $b_\lambda < 0$, for $|\lambda| = n, \lambda_4 = \ell$.

Some properties of A-patches.

a. *For a three-sided j-patch, if $b_\lambda = 0$ for $\lambda = (n-\ell)e_m + \ell e_j$, $\ell = 0, 1, \ldots, k (m \neq j, k < n)$, and $b_\lambda \neq 0$ for $\lambda = (n-1)e_m + e_s$, $s \neq j, m$, then the edge $[e_je_m]$ is tangent with S_F at e_m with multiplicities k. See also Figure 2 (a).*

b. *For a four-sided ij-kℓ-patch, if $b_\lambda = 0$ for $\lambda = (n - q_1 - q_2)e_k + q_1e_i + q_2e_j$, $q_1 + q_2 = 0, 1, \ldots, s$; and $b_\lambda \neq 0$ for $\lambda = (n-1)e_k + e_\ell$, then S_F is tangent s times with face $[e_ie_je_k]$ at e_k.*

Note that a four sided patch may degenerate into a two sided patch. See Figure 2 (b). However, we do not need to treat the degenerate patches any different and consider it to be a special four sided patch.

c. *For a three-sided j-patch, if $b_\lambda = 0$ for $\lambda = (n-m)e_i + me_k$, $m = 0, 1, \ldots, n$, then S_F contains the edge $[e_i, e_k]$. If further, $b_\lambda = 0$, for $\lambda = (n-m-1)e_i + me_k + e_j$, $m = 0, 1, \ldots, n-1$, then the S_F is tangent with the face $[e_ie_je_k]$. See also Figure 3 (a), (b).*

4 Normals and the Simplicial Hull

For the given point set $P = \{p_1, \ldots, p_k\} \in \mathbb{R}^3$ and their surface triangulation T, we first construct a normal set $N = \{n_1, \ldots, n_k\} \in \mathbb{R}^3$ for P. That is, for each point p_i, we associate a normal n_i. We will force the constructed surface to interpolate points p_i and at each point have a normal n_i for $i = 1, \cdots, k$. These normals therefore also provide a mechanism to control the shape of the C^1 interpolating surface. Common approaches to construct these normals at a point p_i include (a) an average of the face normals of the incident faces (b) the gradient of a local spherical fit to the surface triangulation at each vertex. Computing an optimal normal assignment is yet an unsolved problem and we are experimenting with different local and global normal selections schemes [1, 15, 13]. Of course at times the data set can have prespecified normals and this too can be the input of the C^1 fitting algorithm.

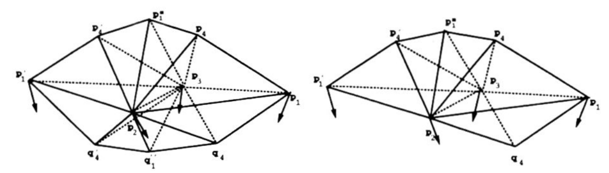


Figure 4: The Construction of Tetrahedra for Adjacent Non-Convex/Non-Convex Faces and Convex/Non-Convex Faces

Without loss of generality we assume that the assigned normals all point to the same side of T. If T is a closed surface triangulation (a simplicial polyhedron) then we assume the normals all point to the exterior.

Definition 4.1. *Convex edge, non-convex edge.*

Let $[p_ip_j]$ be an edge of T. If $(p_j - p_i)^T n_i \; (p_i - p_j)^T n_j \geq 0$ and at least one of $(p_j - p_i)^T n_i$ and $(p_i - p_j)^T n_j$ is positive, then we say the edge $[p_ip_j]$ is *positive convex*. If both the numbers are zero then we say it is *zero convex*. A *negative convex* edge is similarly defined. If $(p_j - p_i)^T n_i \; (p_i - p_j)^T n_j < 0$, then we say the edge is *non-convex*.

Definition 4.2. *Convex face, non-convex face.*

Let $[p_ip_jp_k]$ be a face of T. If its three edges are nonnegative (positive or zero) convex and at least one of them is positive convex, then we say the face $[p_ip_jp_k]$ is *positive convex*. If all the three edges are zero convex then we label the face as *zero convex*. A *negative convex* face is similarly defined. All the other cases $[p_ip_jp_k]$ are labeled as *non-convex*.

Note, that here we are overloading the term *convex* to characterize the relations between the normals and edges of faces. We distinguish between convex and non-convex faces in the simplicial hull below where we build one tetrahedron for convex faces and double tetrahedra for non-convex faces.

Definition 4.3. *Simplicial hull.*

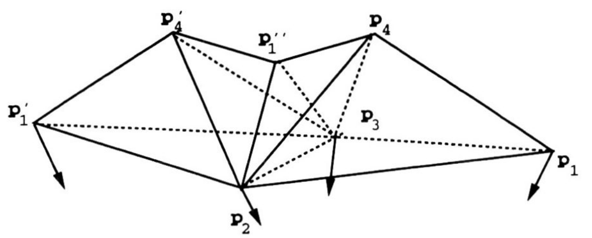


Figure 5: The Construction of Tetrahedra for Adjacent Convex/Convex Faces

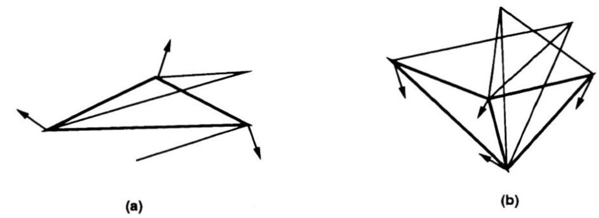


Figure 6: (a) No Tangent Plane Containment (b) Self-Intersecting Tetrahedra

A simplicial hull of T, denoted by $\sum$, is a collection of non-degenerate tetrahedra which satisfies:

(1) Each tetrahedron in $\sum$ has either a single edge of T(then it will be called an *edge tetrahedron*) or a single face of T(then it will be called a *face tetrahedron*).

(2) For each face of T there is/are only one/two face tetrahedron/tetrahedra in $\sum$ if the face is convex/non-convex.

(3) Two face tetrahedra that share a common edge do not intersect anywhere else. This condition is referred to in this paper as *non-intersection*.

(4) For each edge there is/are only one/two pair/pairs of common face sharing edge tetrahedra in $\sum$ if the edge is convex/non-convex such that the pair/pairs fill(s) the region between the two adjacent face tetrahedra in the same side of T.

(5) For each vertex, the tangent plane defined by the vertex normal is contained in all the tetrahedra containing the vertex. This condition is called *tangent plane containment.*

It should be noted that, for a given surface triangulation and normal assignment T, there may exist infinitely many simplicial hulls or no simplicial hull at all. We now describe a scheme for constructing a simplicial hull for the surface triangulation T and prescribed vertex normal assignment. In the full version of the paper [3] we also enumerate the exceptional configurations where a simplicial hull of T is not possible and then provide a solution for constructing the simplicial hull for a locally modified T.

1. Build Face Tetrahedra. For each face $F = [p_1p_2p_3]$ of T, let L be a straight line that is perpendicular to the face F and passes through the center of the inscribed circle of F. Then choose points p_4 and/or q_4 off each side of F to be the farthermost intersection points between L and the tangent planes of the vertices of the face. If F is a non-convex face, two face tetrahedra $[p_1p_2p_3p_4]$ and $[p_1p_2p_3q_4]$ are formed. If F is positive convex, then p_4 is chosen on the side opposite to the direction of the normals, and a single face tetrahedron $[p_1p_2p_3p_4]$ is formed. If F is negative convex, then q_4 is chosen on the same side as the normals and again the single face tetrahedron $[p_1p_2p_3q_4]$ is formed. Figure 5 shows the case where both faces are convex and Figure 4 shows the cases where at least one of the two adjacent faces is non-convex.

A sufficient condition for constructing face tetrahedra with tangent plane containment is that the angle of the assigned normal n_i at each vertex p_i with each of the surrounding face's normals is less than $\pi/2$. If this condition is not met then an exception may occur and we term the vertex as *sharp*. See Figure 6 (a).

A sufficient condition for adjacent face tetrahedra to be non-intersecting is as follows. For two adjacent faces $F = [p_1p_2p_3]$ and $F' = [p_1'p_2p_3]$, the angle between them, denoted as $\angle FF'$, is defined as the outer dihedral angle if the edge between F and F' is negatively convex or inner dihedral angle otherwise. For $[p_2p_3]$, the common edge between F and F', let $[p_1p_2p_3p_4]$ and $[p_1'p_2p_3p_4']$ be the face tetrahedra respectively. Then the two tetrahedra are non-intersecting if the angles $\angle[p_4p_2p_3][p_1p_2p_3] < \frac{1}{2}\angle FF'$ and $\angle[p_4'p_2p_3][p_1'p_2p_3] < \frac{1}{2}\angle FF'$. If this condition is not met then an exception may occur and we term the common edge $[p_2p_3]$ as *sharp*. See Figure 6 (b).

A heuristic strategy rectifies the *sharp* edge and *sharp* vertex configurations is a local retriangulation of the original surface triangulation T. This strategy has worked well in several of the smoothing examples we have performed [3]

2. Build Edge Tetrahedra. Let $[p_2p_3]$ be an edge of T and $[p_1p_2p_3]$ and $[p_1'p_2p_3]$ be the two adjacent faces. Let $[p_1p_2p_3p_4]$ and/or $[p_1p_2p_3q_4]$, and $[p_1'p_2p_3p_4']$ and/or $[p_1'p_2p_3q_4']$ be the face tetrahedra built for the faces $[p_1p_2p_3]$ and $[p_1'p_2p_3]$, respectively. Then if the edge $[p_2p_3]$ is non-convex, two pairs of tetrahedra need to be constructed. The first pair $[p_1''p_2p_3p_4]$ and $[p_1''p_2p_3p_4']$ are between $[p_1'p_2p_3p_4']$ and $[p_1p_2p_3p_4]$. The second pair $[q_1''p_2p_3q_4]$ and $[q_1''p_2p_3q_4']$ are between $[p_1'p_2p_3q_4']$ and $[p_1p_2p_3q_4]$. Here $p_1'' \in (p_4p_4')$ or is above (p_4, p_4'), say

$$p_1'' = \frac{(1-t)}{2}(p_2 + p_3) + \frac{t}{2}(p_4' + p_4), \quad t \geq 1$$

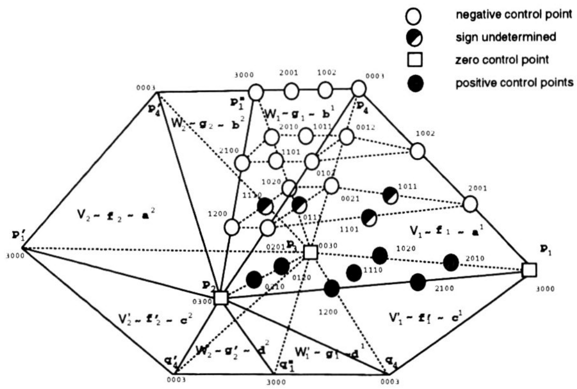


Figure 7: Adjacent Tetrahedra, Functions and Control Points for Two Non-Convex Adjacent Faces

so that p_1'' is above plane $[p_1p_2p_3]$ and plane $[p_1'p_2p_3]$. Similarly, $q_1'' \in (q_4q_4')$ or is below (q_4, q_4'), say

$$q_1'' = \frac{(1-t)}{2}(p_2 + p_3) + \frac{t}{2}(q_4' + q_4), \ t \geq 1$$

so that q_1'' is below plane $[p_1p_2p_3]$ and plane $[p_1'p_2p_3]$. If the edge $[p_2p_3]$ is positive/negative convex, only the first/second pair above are needed. If the edge $[p_2p_3]$ is zero convex, no tetrahedron is needed here. It should be noted that p_4 and p_4'(q_4 and q_4') are always visible.

5 Construction of a C^1 Interpolatory Surface

Having established a simplicial hull $\sum$ for the given surface triangulation T and a set of vertex normals N, we now construct a C^1 function f on the hull $\sum$ such that

$$f(p_i) = 0, \quad \nabla f(p_i) = n_i, \quad i = 1, 2, \ldots, k \qquad (9)$$

and the zero contour of f within $\sum$ forms a C^1 continuous single sheeted surface with the same topology as T.

The construction of the function f over two adjacent faces of T is divided into the following four cases:

(a). Adjacent faces are non-convex;

(b). Adjacent faces are convex;

(c). One face is convex and the other is non-convex;

(d). Adjacent faces are coplanar.

A detailed listing of the specific C^1-conditions and the proofs of solvability of the related linear systems for each of the above cases are given in the full version of the paper [3].

Having built C^1 cubics with some free control points, we now illustrate how to determine these free control points such that the zero-contours are three-sided or four-sided A-patches (smooth and single sheeted).

We assume (without loss of generality) that all the normals point to the same side of the surface triangulation T. That is the side on which q_4 and q_4' lie(see Figure 7). Under this assumption, it follows from Definition 4.1 and equations (2) and (9) that, the control points on the edge, say a^i_{0210}, a^i_{0120} on edge $[p_2p_3]$(see Figure 7), are non-negative if the edge is non-negative convex, and non-positive if the edge is non-positive convex. Now we can divide all the control points into 7 groups called layers. The 0-th layer consists of the control points that are "on" the faces of T. The 1st layer is next to the 0-th layer but opposite to the normal direction, followed by the 2nd and 3rd layers. Next to the 0-th layer and on the same side as the normal, is the -1st layer, then the -2nd and -3rd layers. Now we show that, we can set all the control points on the 2nd and 3rd layer negative and the control points on the -2nd and -3rd layers positive.

For the face-tetrahedra, it is always possible to make the 2nd and 3rd layers control points negative, because these control points are free under the C^0 condition. For the control points on the edge-tetrahedra, it follows from (4) that the 2nd and 3rd layers control points can be negative only if the 2nd layer control points on the neighbor face-tetrahedra are small enough. (See [3] for details.) Similarly, the control points on the -2nd and -3rd layers can be chosen to be positive. Furthermore, all these control points can be chosen as large as one needs in absolute value in order to get single sheeted patches.

Since the control points around the vertices of T are determined by the normals, the smooth vertex condition is obviously satisfied. If the surface contains the edge $[p_2p_3]$(see Figure 7), then since a^i_{1110}(or a^i_{0111}) is freely chosen, the smooth edge condition is easily satisfied(see the proof of Proposition 5.3 in [3]). Referring to Figure 5.1, we prove in the following that the patches constructed over V_1 and W_1 are single sheeted. The other patches are similar.

Proposition 5.2. *If the face $[p_1p_2p_3]$ is non-negative convex, then the control points can be determined so that the surface over V_1 is a three-sided 4-patch.*

Proposition 5.3. *If the edge $[p_2p_3]$ is non-negative convex, then the control points can be determined such that the surface over W_1 is a four-sided 14-23-patch.*

Subdivision. For any face of $T = [p_1, p_2, p_3]$, if it is non-convex and if the three inner products of the face normal and its three adjacent face normals have different signs, then subdivide the double face tetrahedra into 6 subtetrahedra by adding a vertex at the center w of the face (a Clough-Tocher split). The coefficients are specified as before by regarding w as p_1(see Figure 7).

Proposition 5.4. *If the above subdivision procedure above is performed, then the control points can be chosen so that the surface over V_1 is a three-sided 4-patch, and the surface over W_1 is a four-sided 14-23-patch.*

These propositions guarantee that the surface constructed are single sheeted.

6 Shape Control

From the discussion of §5, there are several parameters that can influence the shape of the constructed C^1 surface. These parameters include (a) the length of the normal if its orientation is fixed, (b) a^i_{1110}, and (c) $a^i_{0102} < 0$, $a^i_{1002} < 0$ $a^i_{0012} < 0$, $a^i_{0003} < 0$ and $b^i_{2001} < 0$ for $i = 1, 2$.

(a). Interactive Shape Control

The influence of the length of a normal at a vertex is as follows: if the normal becomes longer then the surface becomes flatter at this point. Parameter a_{1110} lifts the surface upwards to the top vertex of the tetrahedron, while others push the surface downwards toward the bottom of the tetrahedron. In order to get a desirable surface, one may specify some additional data points in the tetrahedron considered, then approximate these points in the least square sense.

(b). Default Shape Control

Here we only consider the effect of the free parameters, that is, suppose the normals are fixed. The aim of the default choice of these parameters is to avoid producing bumpy surfaces. The commonly used method is to keep the surface patch close to a quadric patch([1, 7]).

By least squares approximation of the coefficients of a quadric ([7]), one can derive that

$$a_{1110} = \frac{1}{4}(a_{1200} + a_{2100} + a_{2010} + a_{1020} + a_{0210} + a_{0120})$$

Using the same idea, the other parameters can also be determined. For example, a_λ for $\lambda_4 > 1$ can be determined by the degree elevation formula

$$a_\lambda = \frac{1}{3}\sum_{i=1}^{4} \lambda_i x_{\lambda - e_i}, \quad |\lambda| = 3, \quad \lambda_4 > 1 \qquad (10)$$

where $x_{\lambda-e_i}$ is the solution of the following equations in the least squares sense

$$a_\lambda = \frac{1}{3}\sum_{i=1}^{4} \lambda_i x_{\lambda - e_i}, \quad |\lambda| = 3, \quad \lambda_4 = 0, 1$$

In the same way, b_{2001} can be determined. Therefore, under the C^1 conditions, we can define two sets of control points $\{a^s_\lambda\}$ and $\{a^q_\lambda\}$ over V_1, where $\{a^s_\lambda\}$ is

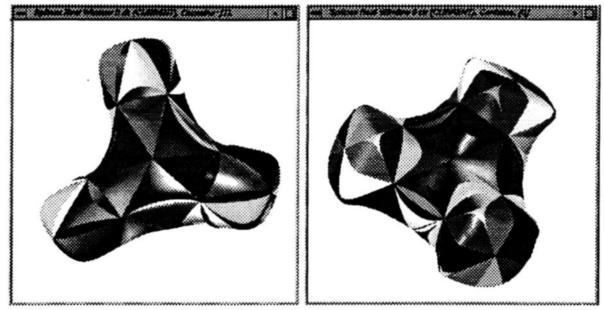


Figure 8: Different Smoothings of the Surface Triangulation Using C^1 Cubic A-Patches

yielded from the single sheeted consideration(see Proposition 5.2–5.6 in [3]), and $\{a^q_\lambda\}$ comes from approximating a simple(quadratic) surface. Note that the surface defined by $\{a^s_\lambda\}$ above may not be desirable in shape, while the surface defined by $\{a^q_\lambda\}$ above may not be single sheeted. In our implementation we take a finite sequence $0 = t_0 < t_1 < \cdots < t_m = 1$ and consider $\{a^{(i)}_\lambda\} = \{(1-t_i)a^q_\lambda + t_i a^s_\lambda\}$, $i = 0, 1, \cdots, m$ selecting the single sheeted surface defined by $\{a^{(i)}_\lambda\}$ for smallest index i. Experiments show that this approach works well and a desirable surface is obtained with $t_i < 0.5$. Examples are shown in Figure 8.

7 Examples

Examples of the simplicial hull construction and C^1 smoothed triangulations using cubic A-patches are shown in Figures 9, 8,and 10. Note in these figures how the "convex" faces are smoothed by a single cubic A-patch per face, while a Clough-Tocher splitting occurs for coplanar faces and some "non-convex" faces, as determined by the vertex normal assignment and the adjacent faces.

References

[1] C. Bajaj. Surface fitting with implicit algebraic surface patches. In H. Hagen, editor, *Topics in Surface Modeling*, pages 23 – 52. SIAM Publications, 1992.

[2] C. Bajaj. The Emergence of Algebraic Curves and Surfaces in Geometric Design. In R. Martin, editor, Directions in Geometric Computing, pages 1 – 29. Information Geometers Press, 1993.

[3] C. Bajaj, J. Chen, and G. Xu. *Modeling with Cubic A-Patches.* Computer Science Technical Report, CAPO-93-02, Purdue University, 1993.

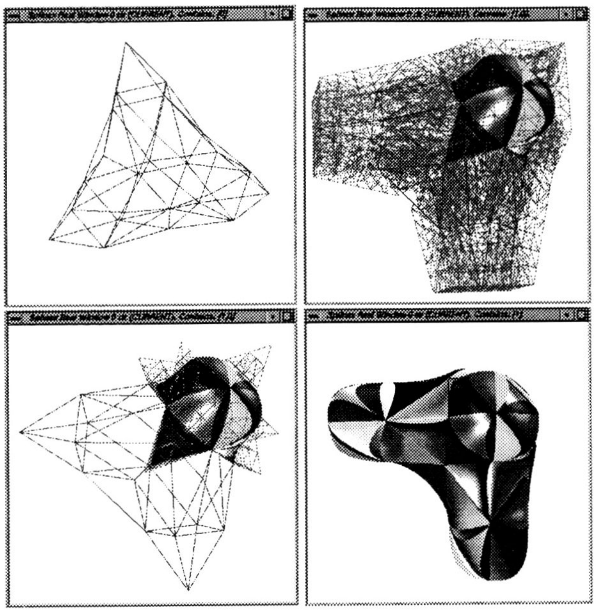


Figure 9: A Surface Triangulation, the Simplicial Hull and some of the interpolatory C^1 Cubic A-Patches

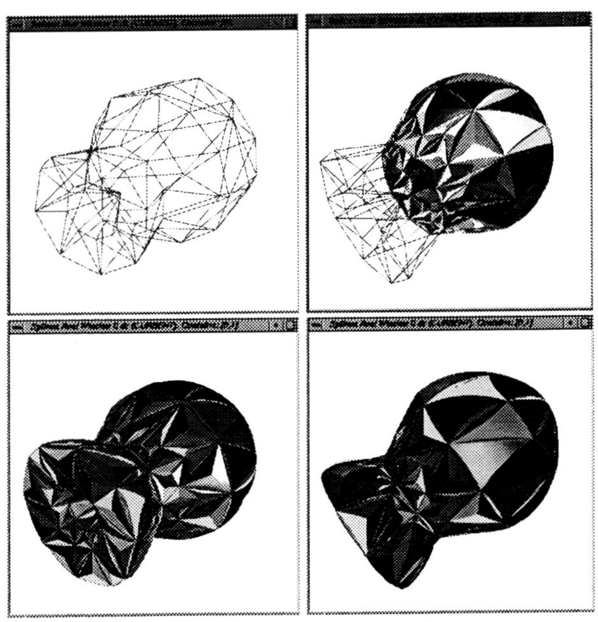


Figure 10: The Smoothing of the Surface Triangulation Using C^1 Cubic A-Patches

[4] C. Bajaj and I. Ihm. C^1 Smoothing of Polyhedra with Implicit Algebraic Splines. *SIGGRAPH'92, Computer Graphics*, 26(2):79–88, 1992.

[5] R. Clough and J. Tocher. Finite Element Stiffness Matrices for Analysis of Plates In Bending. In Proceedings of Conference on Matric Methods in Structural Analysis, 1965.

[6] W. Dahmen. Smooth piecewise quadratic surfaces. In T. Lyche and L. Schumaker, editors, *Mathematical Methods in Computer Aided Geometric Design*, pages 181–193. Academic Press, Boston, 1989.

[7] W. Dahmen and T-M. Thamm-Schaar. Cubicoids: modeling and visualization. *Computer Aided Geometric Design*, 10:93–108, 1993.

[8] G. Farin. *Curves and Surfaces for Computer Aided Geometric Design: A Practical Guide*. Academic Press Inc., 1990.

[9] B. Guo. *Modeling Arbitrary Smooth Objects with Algebraic Surfaces*. PhD thesis, Computer Science, Cornell University, 1991.

[10] B. Guo. Surface generation using implicit cubics. In N.M. Patrikalakis, editor, *Scientific Visualizaton of Physical Phenomena*, pages 485–530. Springer-Verlag,Tokyo, 1991.

[11] S. Lodha. *Surface Approximation with Low Degree Patches with Multiple Representations*. PhD thesis, Computer Science, Rice University, 1992.

[12] D. Moore and J. Warren. Approximation of dense scattered data using algebraic surfaces. In *Proc. of the 24th Hawaii Intl. Conference on System Sciences*, pages 681–690, Kauai, Hawaii, 1991.

[13] H. Moreton. *Minimum Curvature Variation Curve Networks and Surfaces for Fair Free Form Shape Design*. PhD thesis, University of California, Berkeley, 1993.

[14] J. Peters. Local smooth surface interpolation: a classification. *Computer Aided Geometric Design*, 7:191–195, 1990.

[15] H. Pottmann. Interpolation on surfaces using minimum norm networks. *Computer Aided Geometric Design*, 9:51–67, 1992.

[16] T.W. Sederberg. Piecewise algebraic surface patches. *Computer Aided Geometric Design*, 2(1-3):53–59, 1985.