Theoretical Continuity Improvement of Fairing Algorithms

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Abstract
We applied some modification to the original fairing algorithms to improve the final surface quality.

1 Introduction
Scattered data interpolation methods construct surfaces that interpolate locations and first partial derivatives (normals) at the data sites. Often, the data sites are triangulated and spline construction schemes with Bernstein-Bézier triangular patches are used. In general, the minimal degree of Bézier patches required to meet a given order of continuity is high [5]. This degree can be reduced by triangle split schemes. One of the simplest schemes is the Clough-Tocher interpolant [1], which splits each triangle into three smaller ones. This scheme reduces the minimum degree of $C^1$ continuous surfaces from five to three, and has one degree of freedom along each boundary. Kashyap later gave ways to improve the Clough-Tocher interpolants quality by adjusting the available degrees of freedom [4], and in particular by reducing the discontinuity in the crossboundary derivative.

2 Fairing Algorithms
The Clough-Tocher scheme divides each domain triangle into three. Fig. 1 shows the layout of the control points around a cubic exterior boundary, and Fig. 2 shows the layout of the control points around the cubic interior boundaries inside a domain triangle. Fairing algorithms are local optimization algorithms used to improve the surface quality of Clough-Tocher scheme, which includes exterior fairing across the exterior boundaries $(P_1 P_2$ in Fig. 1 and $P Q, Q R, R P$ in Fig. 2) and interior fairing across the interior boundaries $(P_1 S, P_1 S\text{', } P_2 S, P_2 S\text{', } P_3 S, P_3 S\text{'}$ in Fig. 1 and $P S, Q S, R S$ in Fig. 2).

Exterior fairing algorithms modify the values of $l_4$ and $r_4$ in Fig. 1 to minimize the $C^2$ discontinuity while keeping the $C^1$ continuity across the exterior boundary. Interior fairing algorithms modify the values of all triangular control points and $S$ in Fig. 2 to achieve $C^2$ continuity across the interior boundaries. Kashyap applied both fairing algorithms in turn, repeating several times to obtain a new surface. The first surface in Fig. 3 shows an example of the result of this scheme.

However, if we apply these two algorithms, in turn, an infinite number of times, the surface will switch between two stable states instead of one, i.e., the process does not converge to a single surface. Neither the exterior or interior fairing algorithms construct a $C^1$ continuous surface across both interior and exterior boundaries, so one more step of global $C^1$ smoothing is required, which will reduce the improvements of the fairing algorithms.

Since the processes of fairing algorithms contain only linear operations, the process of the algorithms can be represented by matrix operations, and the values of all control points can be stored inside a single column matrix. Then the processes of an algorithm can be described as

\[ v \rightarrow Mv + t, \]

where $M$ and $t$ are constant. Let $f^E(v) = M_{ex}v + t_{ex}$ denote the exterior fairing process and $f^I(v) = M_{in}v + t_{in}$ denote the interior fairing process (with the $C^1$ smoothing: adjust the value of $p_5, q_5, r_5$, and $S$ to meet $C^1$ continuity conditions inside the domain triangles). Then a complete loop can be described as

\[ f(v) = M_{in}(M_{ex}v + t_{ex}) + t_{in}. \]  (1)

If this algorithm converges, with any arbitrary initial value, then the values of the control points will eventually reach a limit $v_1$:

\[ v_1 = M_{in}(M_{ex}v_1 + t_{ex}) + t_{in}. \]

Writing $v_2 = f^E(v_1)$, there exist relationships

\[ v_1 = M_{in}v_2 + t_{in} \quad (a) \]
\[ v_2 = M_{ex}v_1 + t_{ex} \quad (b). \]  (2)

The surface represented by $v_1$ is provided by interior fairing and the surface represented by $v_2$ is provided by exterior fairing. In general the limit of $v_1$ are not equal to the limit of $v_2$, and we need to choose one of them as the final result. However, consider the following proposition:

**Proposition 2.1** Suppose that the repeated application of Equation (1) converges, then Equations (2) hold. In these equations if the subset of the control points updated by (a) in $v_2$ and the subset of the control points updated by (b) in $v_1$ are mutually exclusive, then $v_1 = v_2$.

This means that if each varying control point is updated in only one fairing step, then $v_1 = v_2$. It is possible to do some modification to the interior fairing algorithm, such that there is no overlap between two groups of control points in each step, and the final surface will have the continuity properties of both steps. The simplest way is to directly replace the original interior fairing steps by the $C^1$ smoothing steps.

Figure 1: Layout of the control points around a cubic exterior boundary

![Image of control points](image-url)
Furthermore, there is a quartic version Clough-Tocher scheme, which also provides global $C^1$ continuity surfaces [2]. The quartic exterior fairing algorithm provides $C^2$ continuity across the exterior boundaries. Similar modifications can be applied to this quartic algorithm so that the resulting surface has $C^2$ continuity across the exterior boundaries.

3 Example

Fig. 3 shows the curvature plots of the results provided by the original and modified cubic fairing algorithms, and a quartic surface provided by modified quartic fairing algorithms which has $C^2$ continuous exterior boundaries. The input data sampled from a $4 \times 4$ grid over the domain $[0, 1] \times [0, 1]$ of the Franke’s function No.1 [3]

$$F_1(x,y) = 0.75\exp\left(-\frac{(9x-2)^2}{4} - \frac{(9y-2)^2}{4}\right) + 0.75\exp\left(-\frac{(9x+1)^2}{49} - \frac{(9y+1)^2}{10}\right) + 0.5\exp\left(-\frac{(9x-7)^2}{4} - \frac{(9y-3)^2}{4}\right) - 0.2\exp\left(-\frac{(9x-4)^2}{4} - \frac{(9y-7)^2}{4}\right).$$

Both results are globally $C^1$ continuity, but the second one has lower curvature discontinuity.

References


