GEOMETRIC CONTINUITY AND SHAPE PARAMETERS FOR CATMULL-ROM SPLINES

(Extended Abstract)

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ABSTRACT

Polynomial Catmull-Rom splines have local control, can be either approximating or interpolating, and are efficiently computable. Practical experience with Beta-splines has shown that it is useful to endow a spline with *shape parameters*, used to modify the shape of the curve or surface independent of the defining control vertices. Thus, it is desirable to construct a subclass of the Catmull-Rom splines which has shape parameters.

We present such a class, some members of which are interpolating and others approximating. As was done for the Beta-spline, shape parameters are introduced by requiring *geometric* rather than *algebraic* continuity. Splines in this class are defined by a set of control vertices and a set of shape parameter values associated with the joints of the curve. The shape parameters may be applied globally, affecting the entire curve, or they may be modified locally, affecting only a portion of the curve or surface near the corresponding joint. The interpolating members of the class are new in that no previous local interpolating technique possesses locally variable shape parameters.

We show that this class results from combining geometric continuous (Beta-spline) blending functions with a new set of geometric continuous interpolating functions. The interpolating functions are shown to be a geometric continuous generalization of the classical *Lagrange polynomials*.

KEYWORDS: Curves and surfaces, Computer-aided geometric design.

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1. Introduction

Many applications of computer-aided geometric design require the description of objects using mathematical functions called *splines*. A spline curve is a piecewise univariate function that satisfies a set of *continuity constraints* where the curve *segments* meet. The point at which two segments join is called a *joint*. A popular type of spline is the *polynomial* spline, defined by a set of *control vertices* and a set of polynomial functions called *basis functions* that are used to blend, or weight, the vertices.

Splines are either *interpolating* or *approximating*. Interpolating splines are required to pass through the control vertices, while approximating splines are only required to pass "near" the vertices. Splines can be further classified as either *global*, or *local* representations. In a global representation, the movement of a control vertex causes the entire spline to change. In a local representation, it is possible to localize the change resulting from the perturbation of a control vertex; this is the property of *local control*. The recent development of the *Beta-spline*[1,2,3,5] has shown that it is possible to introduce *shape parameters* into the curve formulation, which can be used to modify the shape of the curve independent of the control vertices. Experience has shown that shape parameters provide a designer with intuitive control of shape.

From the standpoint of computer-aided geometric design, it is desirable to construct local, polynomial splines with shape parameters. Since the choice of interpolation versus approximation is application dependent, both should be possible. The objective of this work is to develop a class of splines possessing shape parameters that are local, polynomial, and either interpolating or approximating. This can be done by combining the work of Catmull and Rom[7] with that of Barsky.[3]

Catmull and Rom[7] introduced a class of local polynomial splines which could be made to either interpolate or approximate a set of control vertices.† To construct a class of splines with the properties enumerated above, we need only introduce shape parameters into the Catmull-Rom splines. As with Beta-splines, this is done by replacing *algebraic* continuity with *geometric* continuity.

Algebraic continuity refers to the continuity of parametric derivative vectors of the curve. A continuous first derivative vector gives *first order algebraic*, or C^1 continuity. If both the first and second derivative vectors are continuous, the spline has *second order algebraic* (C^2) continuity. Geometric continuity, on the other hand, requires continuity of visual quantities such as *unit tangent* and *curvature* vectors. A continuous unit tangent vector gives *first order geometric* (G^1) continuity, while *second order geometric* (G^2) continuity refers to continuous unit tangent and curvature vectors.

It has previously been shown[3] that C^1 continuity may be replaced with G^1, and C^2 may be replaced with G^2 while still maintaining visual smoothness. Since geometric continuity is less restrictive than the corresponding order of algebraic continuity, the relaxation from algebraic to geometric continuity allows the introduction of new degrees of freedom called *shape parameters*. The replacement of C^1 with G^1 results in one shape parameter; replacing C^2 with G^2 results in two shape parameters.

† Unfortunately, the title of their paper did not reflect the fact that both approximating and interpolating splines are members of the class.

This paper shows how the relaxation to geometric continuity can yield a class of Catmull-Rom splines, either interpolating or approximating, whose shape can be modified via shape parameters. The interpolating splines we present are new due to their shape parameters; they are the first local, polynomial, interpolating splines with locally variable shape parameters. Consequently, local modification of a shape parameter affects only a portion of the curve near the corresponding joint.

2. Notation

Scalar quantities will be written in Italics (x, $Y(u)$), and vectors and vector-valued functions will be denoted by boldface type ($\mathbf{V}$, $\mathbf{Q}(u)$). Since it is often necessary to distinguish between a piecewise function and the segments that compose it, we will adhere to the convention that a piecewise function is denoted by an upper case character ($\Lambda(u)$, $\mathbf{H}_q(u)$), while the segments are indexed and written in the corresponding lower case ($\lambda_i(u)$, $\mathbf{h}_{q,j}(u)$). Finally, the p^{th} derivative of a parametric function, when taken with respect to its domain parameter, will be denoted with a superscript (p) as in $W^{(p)}(u)$ and $\mathbf{Q}^{(p)}(u)$.

3. The Class of Catmull-Rom Splines

Splines used in computer-aided geometric design are typically defined by a set of *control vertices* $\mathbf{V}_i$ and a set of blending functions $W_i(u)$; i.e.

$$\mathbf{Q}(u) = \sum_{i=0}^{m} \mathbf{V}_i W_i(u) \qquad (3.1)$$

Catmull and Rom extended this form by replacing the vertices $\mathbf{V}_i$ with vector-valued *interpolating functions* $\mathbf{P}_i(u)$. Each $\mathbf{P}_i(u)$ is constructed to interpolate the $k+1$ vertices $\mathbf{V}_i, \mathbf{V}_{i+1}, \cdots, \mathbf{V}_{i+k}$, for some nonnegative integer k. Intuitively, k sets the width of the *interpolating window* of the function $\mathbf{P}_i(u)$. Thus, a Catmull-Rom spline takes the form

$$\mathbf{F}(u) = \sum_{i=0}^{m} \mathbf{P}_i(u) W_i(u) \qquad (3.2)$$

Catmull and Rom show that if the blending functions are non-zero over D parametric intervals, then a spline of the form given in (3.2) will be approximating if $k < D-2$, and interpolating otherwise. When $k=0$, the function $\mathbf{P}_i(u)$ is only required to interpolate $\mathbf{V}_i$, so it is sufficient for $\mathbf{P}_i(u) = \mathbf{V}_i$. In this case, equation (3.2) is identical in form to equation (3.1), showing that the Catmull-Rom splines generalize standard approximating techniques such as Bézier,[6] B-splines,[10] and Beta-splines.[3]

Throughout the remainder of our discussion, we make the following assumptions:

- The q^{th} segment of $\mathbf{F}(u)$, denoted $\mathbf{f}_q(u)$, is traced out when u is on the half-open interval $[q, q+1)$ (see figure 3.1).

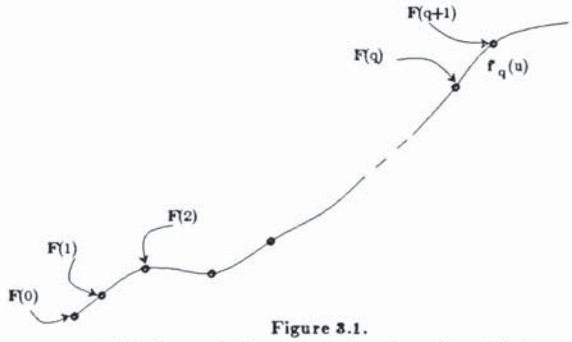

Figure 3.1.
Indexing of the piecewise function $\mathbf{F}(u)$. *Note that the joints correspond to integral values of the domain parameter.*

- The blending functions have *local support*; that is, they are nonzero over only over a parametric width of D. The i^{th} such function $W_i(u)$ is nonzero only over the open interval $(i-1, i-1+D)$.
- The blending functions form a *partition of unity*; that is, they satisfy

$$\sum_{i=0}^{m} W_i(u) = 1 \quad \text{for } 0 \le u < m \tag{3.3}$$

- Finally, the interpolating functions $\mathbf{P}_i(u)$ are constructed so that

$$\mathbf{P}_i(q) = \mathbf{V}_q \quad \text{for } q = i, i+1, \cdots, i+k \tag{3.4}$$

Since the blending functions have local support, the q^{th} segment of $\mathbf{F}(u)$ can be written as

$$\mathbf{f}_q(u) = \sum_{i=q+2-D}^{q+1} \mathbf{P}_i(u) W_i(u) \quad q \le u < q+1 \tag{3.5}$$

It is convenient to change the parametrization of $\mathbf{f}_q(u)$ such that it is traced out when its parameter varies on the interval [0,1). To do this, we can without loss of generality assume that the blending functions $W_i(u)$ and interpolating functions $\mathbf{P}_i(u)$ are piecewise functions where each of the segments is parametrized on [0,1) (see figure 3.2). If the j^{th} segments of $W_i(u)$ and $\mathbf{P}_i(u)$ are denoted by $w_{i,j}(u)$ and $\mathbf{p}_{i,j}(u)$, respectively, then a [0,1) parametrization for $\mathbf{f}_q(u)$ is

$$\mathbf{f}_q(u) = \sum_{i=0}^{D-1} \mathbf{p}_{q+i+2-D,i}(u)\; w_{q+i+2-D,i}(u) \quad 0 \le u < 1 \tag{3.6}$$

4. Geometric Continuity

Consider the situation at the joint where the segments $\mathbf{q}_{i-1}(u)$ and $\mathbf{q}_i(u)$ meet (see figure 4.1). The necessary and sufficient condition for unit tangent vector continuity at the joint between $\mathbf{q}_{i-1}(u)$ and $\mathbf{q}_i(u)$ is[3]

$$\mathbf{q}_i^{(1)}(0) = \beta 1_i \mathbf{q}_{i-1}^{(1)}(1) \tag{4.1}$$

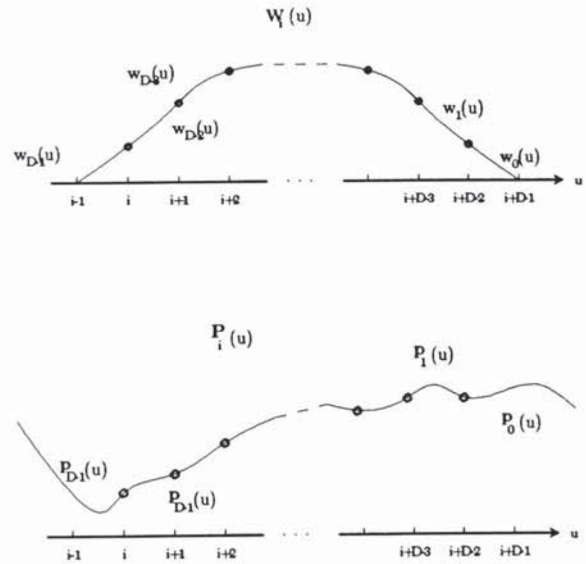

Figure 3.2.
The j^{th} *segments of* $W_i(u)$ *and* $\mathbf{P}_i(u)$ *are denoted by* $w_{i,j}(u)$ *and* $\mathbf{p}_{i,j}(u)$, *respectively. They are indexed from right to left, as shown above.*

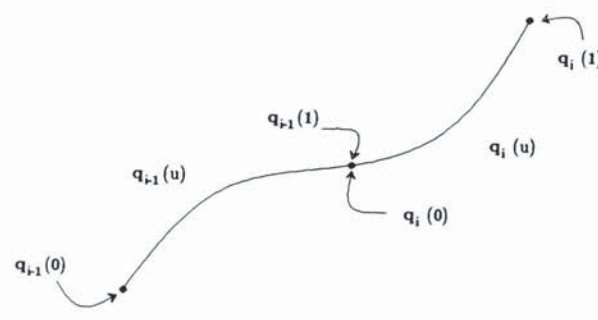

Figure 4.1.
The joint common to the segments $\mathbf{q}_{i-1}(u)$ *and* $\mathbf{q}_i(u)$ *corresponds to* $\mathbf{q}_{i-1}(1)$ *and* $\mathbf{q}_i(0)$.

This constraint must hold for all values of $\beta 1_i$. The condition for C^1 continuity is a special case of (4.1), occurring when $\beta 1_i = 1$. If $\beta 1_i \ne 1$, equation (4.1) prescribes a discontinuity in the first parametric derivative. However, due to the parametric representation of $\mathbf{q}_{i-1}(u)$ and $\mathbf{q}_i(u)$, this discontinuity does not affect the visual smoothness at the joint.[3]

Recall that a uniform quadratic B-spline blending function is C^1. The segments of the blending function are given by

$$n_0(u) = \frac{u^2-2u+1}{2}$$
$$n_1(u) = \frac{-2u^2+2u+1}{2} \qquad (4.2)$$
$$n_2(u) = \frac{u^2}{2}$$

where each segment is parametrized on [0,1) and the indexing is from right to left, as in figure 3.2. The G^1 analogue of the uniform quadratic B-spline is the G^1 Beta-spline, the segments of which are given by[9]

$$b_{i,0}(u) = \frac{\beta 1_{i+1}}{\beta 1_{i+1}+1}\left(u^2-2u+1\right)$$

$$b_{i,1}(u) = (-(\beta 1_i \beta 1_{i+1} + 2\beta 1_i + 1)u^2 + 2\beta 1_i(\beta 1_{i+1}+1)u - \beta 1_{i+1}+1))/((\beta 1_i+1)(\beta 1_{i+1}+1)) \qquad (4.3)$$

$$b_{i,2}(u) = \frac{u^2}{\beta 1_i + 1}$$

Second order geometric continuity (G^2) requires not only continuity of the unit tangent vector, but continuity of the curvature vector as well. Thus, for G^2 continuity, equation (4.1) must hold in addition to[3]

$$\mathbf{q}_i^{(2)}(0) = \beta 1_i^2 \mathbf{q}_{i-1}^{(2)}(1) + \beta 2_i \mathbf{q}_{i-1}^{(1)}(1) \qquad (4.4)$$

where $\beta 1_i$ and $\beta 2_i$ can be freely chosen. G^2 reduces to C^2 when $\beta 1_i = 1$ and $\beta 2_i = 0$. When $\beta 1_i$ and $\beta 2_i$ do not have these *default values*, equations (4.1) and (4.4) specify discontinuities in the first and second parametric derivatives, respectively.

Just as the G^1 Beta-spline is the geometric analogue of the uniform quadratic B-spline, the G^2 Beta-spline is the geometric analogue of the uniform cubic B-spline. The segments of the blending functions are fairly complex, but can be found in Goodman.[9]

5. Geometric Continuous Catmull-Rom Splines

We now apply the notion of geometric continuity to the Catmull-Rom spline. The resulting subclass can conveniently be described by Table 5.1.

window width	Continuity Constraint	
	G^1	G^2
$k = 0$	Approximating •	Approximating
$k = 1$	Interpolating •	Approximating
$k > 1$	Interpolating	Interpolating

• Explicitly constructed in section 6

Table 5.1

The rows of the table correspond to the width of the interpolating window (k) used in the construction of the function $\mathbf{P}_i(u)$ in equation (3.5). Since the splines in the first column are G^1 continuous, they possess one shape parameter per joint; the splines in the second column have two shape parameters per joint.

5.1. Decoupling

To construct a Catmull-Rom spline $\mathbf{F}(u)$ subject to the first order geometric continuity constraint (4.1), it is sufficient to use blending functions $W_i(u)$ and interpolating functions $\mathbf{P}_i(u)$, each of which separately satisfies equation (4.1). The proof proceeds by differentiation of (3.6), followed by substitution of

$$\begin{aligned} w_{j,i}(0) &= w_{j,i-1}(1) \\ w_{j,i}^{(1)}(0) &= \beta 1_q\, w_{j,i-1}^{(1)}(1) \\ \mathbf{p}_{j,i}(0) &= \mathbf{p}_{j,i-1}(1) \\ \mathbf{p}_{j,i}^{(1)}(0) &= \beta 1_q\, \mathbf{p}_{j,i-1}^{(1)}(1) \end{aligned} \qquad j = q+i+2-D \qquad (5.1)$$

which must hold if $W_i(u)$ and $\mathbf{P}_i(u)$ are G^1 continuous. One can then show that the resulting expression is equivalent to

$$\mathbf{f}_q^{(1)}(0) = \beta 1_q \mathbf{f}_{q-1}^{(1)}(1) \qquad (5.2)$$

Thus, the determination of a first order geometric continuous Catmull-Rom spline decouples into two smaller problems: the construction of G^1 blending functions $W_i(u)$, and the construction of G^1 interpolating functions $\mathbf{P}_i(u)$. Using a similar technique, it is possible to show that G^2 Catmull-Rom splines result from the combination of G^2 blending functions and G^2 interpolating functions.

The geometric continuous blending functions necessary to construct geometric continuous Catmull-Rom splines are known; they are the G^1 and G^2 Beta-spline blending functions.[3, 9] All that remains is the development of G^1 and G^2 interpolating functions $\mathbf{P}_i(u)$.

5.2. Geometric Continuous Interpolating Functions

Before embarking on the derivation of the geometric continuous functions $\mathbf{P}_i(u)$, we examine the functions originally used by Catmull and Rom to show that they are not geometric continuous, and hence cannot be used to construct a geometric continuous Catmull-Rom spline. However, a generalization of the functions they chose can be used.

5.2.1. Lagrange Interpolation

Recall that the function $\mathbf{P}_i(u)$ must be constructed to interpolate the vertices $\mathbf{V}_i, \mathbf{V}_{i+1}, \cdots, \mathbf{V}_{i+k}$, for some nonnegative integer k. Catmull and Rom chose functions of the form

$$\mathbf{P}_i(u) = \sum_{j=0}^{k} \mathbf{V}_{i+j}\, L_j(k; u-i) \qquad (5.3)$$

where the $L_j(k;u)$ are the classical *Lagrange polynomials*, defined by

$$L_j(k;u) = \prod_{\substack{p=0 \\ p \neq j}}^{k} \left(\frac{u-p}{j-p}\right) \qquad (5.4)$$

It is easy to show that the Lagrange polynomials satisfy the *Kronecker delta relation*

$$L_j(k;r) = \delta_{j,r} = \begin{cases} 0 \text{ for } r \neq j \\ 1 \text{ for } r = j \end{cases} \quad \text{for } r = 0,1,\cdots,k \quad (5.5)$$

In fact, any set of functions satisfying equation (5.5) can be used to construct an interpolating function of the form given in (5.3). We now examine the continuity of $\mathbf{P}_i(u)$.

A function $\mathbf{P}_i(u)$ as in equation (5.3) is a single polynomial of degree k, and as such is everywhere C^∞ continuous. It is impossible for such a function to have the derivative discontinuities required by the geometric continuity conditions (4.1) and (4.4). Intuitively, the Lagrange polynomials are *too smooth* to be geometric continuous. For our purposes, we must construct a set of functions that satisfy both a Kronecker delta relation similar to (5.5) and either first or second order geometric continuity. We do this by developing a set of piecewise polynomials that can be shown to be a generalization of the Lagrange polynomials presented above.

5.2.2. Geometric Continuous Lagrange Interpolation

Since a single polynomial is everywhere C^∞, we must resort to a piecewise polynomial representation to obtain interpolating functions that have the derivative discontinuities necessary for geometric continuity. We choose functions of the same form as (5.3), but we replace the Lagrange polynomials with a set of piecewise functions $\Lambda_{i,j}(k;u)$ to get

$$\mathbf{P}_i(u) = \sum_{j=0}^{k} \mathbf{V}_{i+j}\Lambda_{i,j}(k;u) \quad (5.6)$$

The properties of $\Lambda_{i,j}(k;u)$ determine the behaviour of $\mathbf{P}_i(u)$. For $\mathbf{P}_i(u)$ to interpolate $\mathbf{V}_i, \mathbf{V}_{i+1}, \cdots, \mathbf{V}_{i+k}$, the functions $\Lambda_{i,j}(k;u)$ must satisfy the Kronecker delta relation

$$\Lambda_{i,j}(k;r) = \delta_{j,r} \quad \text{for } r = i, i+1, \cdots, i+k \quad (5.7)$$

Moreover, since the continuity of $\Lambda_{i,j}(k;u)$ is inherited by $\mathbf{P}_i(u)$, $\Lambda_{i,j}(k;u)$ must be constructed to possess geometric continuity at its joints.

Let $\lambda_{i,j,s}(k;u)$, $j = 0,1,\cdots,k$, and $s = 0,1,\cdots,D-1$ be the s^{th} segment of $\Lambda_{i,j}(k;u)$, as shown in figure 5.1. For convenience in developing a [0,1) parametrization for $\mathbf{f}_q(u)$, we require each of the $\lambda_{i,j,s}(k;u)$ to be parametrized on [0,1).

Observe that $\Lambda_{i,j}(k;u)$ requires D segments, where D is the width of the blending function $W_i(u)$ that will ultimately weight it. Any additional segments of $\Lambda_{i,j}(k;u)$ would not contribute to (3.2) when weighted by $W_i(u)$. If $\Lambda_{i,j}(k;u)$ had fewer segments, it would not be geometric continuous at each of the joints within the support of $W_i(u)$.

Although the Lagrange polynomials have an elegant, concise definition (equation (5.4)), we do not know of a closed form for the functions $\Lambda_{i,j}(k;u)$ for arbitrary k and D; they must currently be constructed on a case-by-case basis for a given k and D. We demonstrate their construction by the following example.

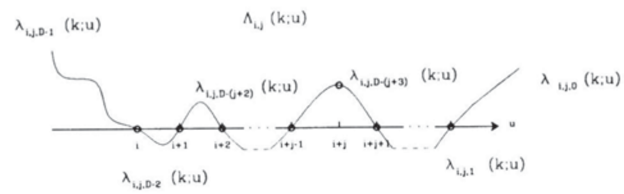

Figure 5.1.
The function $\Lambda_{i,j}(k;u)$ has segments $\lambda_{i,j,s}(k;u)$, indexed from right to left, as shown above.

Consider the case where it is desired to construct G^1 continuous $\Lambda_{i,j}(k;u)$ for $k = 1$ and $D = 3$. The six functions $\lambda_{i,j,s}(1;u)$, $j = 0,1$, $s = 0,1,2$ must be determined subject to the Kronecker delta constraint and first order geometric continuity. The Kronecker delta constraints for the segments of $\Lambda_{i,0}(1;u)$ are (see figure 5.2).

$$\begin{aligned} \lambda_{i,0,2}(1;1) &= 1 \\ \lambda_{i,0,1}(1;0) &= 1 \\ \lambda_{i,0,0}(1;1) &= 0 \\ \lambda_{i,0,0}(1;0) &= 0 \end{aligned} \quad (5.8)$$

and the G^1 constraints are

$$\begin{aligned} \lambda^{(1)}_{i,0,0}(1;0) &= \beta 1_{i+1}\lambda^{(1)}_{i,0,1}(1;1) \\ \lambda^{(1)}_{i,0,1}(1;0) &= \beta 1_i \lambda^{(1)}_{i,0,2}(1;1) \end{aligned} \quad (5.9)$$

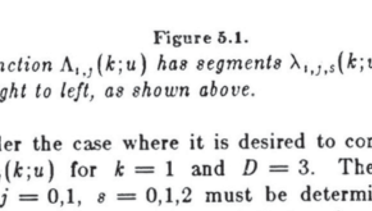

Figure 5.2.
The position and indexing of the segments of $\Lambda_{i,0}(1;u)$

This set of equations imposes six constraints on the three functions $\lambda_{i,0,0}(1;u)$, $\lambda_{i,0,1}(1;u)$, and $\lambda_{i,0,2}(1;u)$. If each function is written in terms of two unknowns, the system will be completely specified. In other words, it is sufficient for each of the functions to be a polynomial of first degree. Assuming they are linear functions, solution of the system yields

$$\begin{aligned} \lambda_{i,0,0}(1;u) &= -\beta 1_{i+1} u \\ \lambda_{i,0,1}(1;u) &= 1-u \\ \lambda_{i,0,2}(1;u) &= \frac{\beta 1_i + 1-u}{\beta 1_i} \end{aligned} \quad (5.10)$$

A similar system of equations can be generated and solved to produce the components of $\Lambda_{i,1}(1;u)$:

$$\begin{aligned}\lambda_{i,1,0}(1;u) &= \beta 1_{i+1} u + 1 \\ \lambda_{i,1,1}(1;u) &= u \qquad (5.11) \\ \lambda_{i,1,2}(1;u) &= \frac{u-1}{\beta 1_i}\end{aligned}$$

When $\beta 1_i = 1$ for all i, it is easy to show that $\Lambda_{i,0}(1;u)$ and $\Lambda_{i,1}(1;u)$ are piecewise linear representations of $L_0(1;u-i)$ and $L_1(1;u-i)$, respectively. This verifies that $\Lambda_{i,j}(1;u)$ form a G^1 generalization of the first degree Lagrange polynomials. The analogous G^2 functions $\Lambda_{i,j}(2;u)$ can be constructed by requiring that the twelve functions $\lambda_{i,j,s}(2;u)$, $j = 0,1,2$, $s = 0,1,2,3$ satisfy an appropriate Kronecker delta relation, and the G^1 and G^2 constraints from (4.1) and (4.4).

5.3. The General Form

In this section we merge a set of geometric continuous blending functions $W_i(u)$ with a set of geometric continuous interpolating functions $\mathbf{P}_i(u)$ to produce a geometric continuous Catmull-Rom spline.

We wish to obtain a [0,1) parametrized version of $\mathbf{f}_q(u)$ similar to equation (3.6) using the form of the interpolating functions given in (5.6). To do this, we require a form for the segments of $\mathbf{P}_i(u)$ referred to in (3.6) in terms of the segments of $\Lambda_{i,j}(u)$. The appropriate expression is

$$\mathbf{P}_{q+i+2-D,i}(u) = \sum_{j=0}^{k} \mathbf{V}_{q+i+2-D+j}\ \lambda_{q+i+2-D,j,i}(k;u) \qquad (5.12)$$

By substituting (5.12) into (3.6), expanding the summations, then regrouping terms and changing summation indices, $\mathbf{f}_q(u)$ can be rewritten as

$$\mathbf{f}_q(u) = \sum_{g=2-D}^{k+1} \mathbf{V}_{q+g}\ \phi_{q,g}(u) \qquad (5.13)$$

where

$$\phi_{q,g}(u) = \sum_{h_{\min} \le h \le h_{\max}} \lambda_{a+h,g+D-2-h,h}(k;u)\ w_{a+h,h}(u) \qquad (5.14)$$

and

$$\begin{aligned}a &= q-D+2 \\ h_{\max} &= \min(g+D-2, D-1) \qquad (5.15) \\ h_{\min} &= \max(0, g+D-2-k)\end{aligned}$$

Equations (5.13) and (5.14) together define the class of geometric continuous Catmull-Rom splines. A particular member of the class is determined by the order of geometric continuity, either G^1 or G^2, and the width of the interpolating window (k). The functions $\phi_{q,g}(u)$ are called the *geometric continuous Catmull-Rom basis functions*. Several important properties of the class can be identified:

(1) Every member of the class has local control. From equation (5.13), $\mathbf{f}_q(u)$ depends only on the $k+D$ vertices $\mathbf{V}_{q+2-D}, \mathbf{V}_{q+3-D}, \cdots, \mathbf{V}_{q+k+1}$. Modification of vertices outside this range has no effect on the segment. Thus, perturbation of a given vertex will only affect $k+D$ segments near it.

(2) Every member has shape parameters. The G^1 splines have one shape parameter per joint; the G^2 splines have two shape parameters per joint. Due to the local control property (1), modification of a particular shape parameter affects at most $k+D$ segments of the curve.

(3) Members of this class can be either interpolating or approximating. Since this class is a proper subclass of the Catmull-Rom splines, if $k<D-2$, the spline will approximate the vertices, otherwise it will interpolate the vertices.

(4) If the functions $\lambda_{l,j,i}(k;u)$ and the functions $w_{l+i,i}(u)$ satisfy

$$\sum_{j=0}^{k} \lambda_{l,j,i}(k;u) = 1 \qquad (5.16)$$

and

$$\sum_i w_{l+i,i}(u) = 1 \qquad (5.17)$$

then the resulting Catmull-Rom basis functions $\phi_{q,g}(u)$ will form a partition of unity. This is a necessary property since it guarantees that the spline will be coordinate system independent. The λ functions constructed in section 5.2 and the Beta-spline blending functions satisfy (5.16) and (5.17), respectively. Their combination according to equation (5.14) therefore gives rise to a coordinate system independent Catmull-Rom spline.

The functions appearing on the right of equation (5.14) need not be polynomial, but previous sections have shown examples of polynomials satisfying the necessary constraints. The rest of the discussion will assume a polynomial representation for $\phi_{q,g}(u)$, but it should be noted that the class defined by equations (5.13) and (5.14) is much more general.

If a G^1 Catmull-Rom spline is desired (a member of the first column of Table 5.1), the functions $w_{i,h}(u)$ are the G^1 Beta-spline blending functions $b_{i,h}(u)$ from (4.3), and the λ functions are constructed to possess G^1 continuity subject to the appropriate Kronecker delta relation. To define a G^2 Catmull-Rom spline, the $w_{i,h}(u)$ are the G^2 Beta-spline blending functions, and the λ functions are constructed for G^2 continuity.

6. Examples

We now construct the splines corresponding to the "starred" entries of Table 5.1, thereby demonstrating the use of equations (5.13) and (5.14).

6.1. The $k = 0$ Row

In Section 3, it was shown that when $k = 0$ a Catmull-Rom spline reduces to the standard form for an approximating spline. Thus, the entries in the $k = 0$ row of Table 5.1 are the G^1 and G^2 Beta-splines. The G^1 blending functions were given in equation (4.3) and the G^2 blending functions may be found in Goodman.[9]

6.2. The ($k = 1$, G^1) Spline

The spline in the ($k = 1$, G^1) position of Table 5.1 is constructed by combining the G^1 Beta-spline blending functions from equation (4.3) with the G^1 functions $\Lambda_{i,j}(1;u)$ from equations (5.10) and (5.11). The combination is governed by (5.14) with $D = 3$ and $k = 1$. Since $k = 1 \geq D-2$, the resulting spline will interpolate the control vertices. Substituting $D = 3$ and $k = 1$ into (5.14) yields the following basis functions

$$\phi_{q,-1}(u) = \lambda_{q-1,0,0}(1;u)\; w_{q-1,0}(u) \qquad (6.1)$$

$$= -\frac{\beta 1_q^2 u^3 - 2\beta 1_q^2 + \beta 1_q^2 u}{\beta 1_q + 1}$$

$$\phi_{q,0}(u) = \lambda_{q-1,1,0}(1;u)\; w_{q-1,0}(u) + \lambda_{q,0,1}(1;u)\; w_{q,1}(u) \qquad (6.2)$$

$$= ((\beta 1_q \beta 1_{q+1} + \beta 1_q + 1)u^3 - (2\beta 1_q \beta 1_{q+1} + 2\beta 1_q + 1)u^2 + ((\beta 1_q - 1)\beta 1_{q+1} + \beta 1_q - 1)u + 1)/(\beta 1_{q+1} + 1)$$

$$\phi_{q,1}(u) = \lambda_{q,1,1}(1;u)\; w_{q,1}(u) + \lambda_{q+1,0,2}(1;u)\; w_{q+1,2}(u) \qquad (6.3)$$

$$= -((\beta 1_q \beta 1_{q+1} + \beta 1_q + 1)u^3 - (2\beta 1_q \beta 1_{q+1} + \beta 1_q + 1)u^2 - \beta 1_{q+1} u)/(\beta 1_{q+1}(\beta 1_q + 1))$$

$$\phi_{q,2}(u) = \lambda_{q+1,1,2}(1;u)\; w_{q+1,2}(u) \qquad (6.4)$$

$$= \frac{u^3 - u^2}{\beta 1_{q+1}(\beta 1_q + 1)}$$

Examination of the basis functions reveals that $\phi_{q,-1}(u)$ depends only on $\beta 1_q$; $\phi_{q,0}(u)$ depends only on $\beta 1_q$ and $\beta 1_{q+1}$; $\phi_{q,1}(u)$ depends only on $\beta 1_q$ and $\beta 1_{q+1}$, and $\phi_{q,2}(u)$ depends only on $\beta 1_{q+1}$. Thus, the curve segment $\mathbf{f}_q(u)$ depends only on the shape parameters $\beta 1_q$ and $\beta 1_{q+1}$. In other words, modification of the shape parameter associated with vertex $\mathbf{V}_q$ affects only two segments: $\mathbf{f}_{q-1}(u)$ and $\mathbf{f}_q(u)$. This behaviour is shown in figure 6.1.

7. Conclusion

This paper has introduced a subclass of the Catmull-Rom splines that possesses geometric, rather than algebraic, continuity. The replacement of algebraic continuity with the less restrictive geometric analogue allows the introduction of shape parameters that can be used to modify the shape of the spline without moving the control vertices. There are either one or two shape parameters per joint which can be independently varied to control the shape of the curve. In addition to shape parameters, members of the class have local control. Some of the splines in the class interpolate the control vertices, while others approximate them (see Table 5.1).

The class results from the combination of Beta-spline blending functions and a set of geometric continuous functions related to the classical Lagrange polynomials. The class is a proper generalization of the algebraic-continuous Catmull-Rom splines. Moreover, the class includes the G^1 and G^2 Beta-splines, which are local, approximating, polynomial splines with shape parameters.

The ($k = 1$, G^1) spline, from section 6.2, and the ($k = 2$, G^2) spline are new to computer-aided geometric design; they are local, polynomial, interpolating splines possessing locally variable shape parameters. Previous interpolating splines either had shape parameters, but were global representations,[4,8,11,12,13] or were local with no shape parameters.[7]

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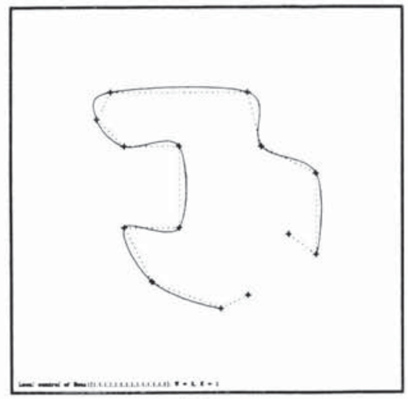
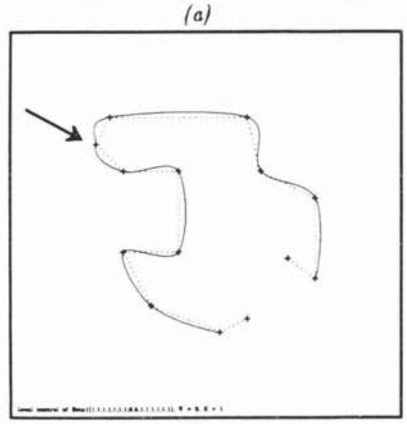
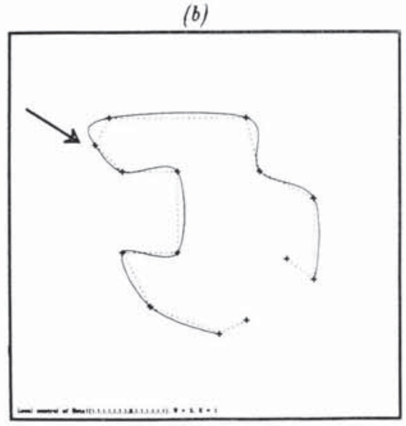


(a)



(b)



(c)

Figure 6.1.
The figures above depict a $(k = 0,\ G^1)$ *geometric continuous Catmull-Rom spline. The value of* $\beta 1$ *associated with each of the joints (or equivalently, the vertices) in (a) is 1. In (b) the value of* $\beta 1$ *for the vertex pointed to by the arrow has been changed to* $1/2$.. *Note that only the two segments adjacent to the vertex have been affected. The value of* $\beta 1$ *in (c) is 2, showing how reciprocal values of* $\beta 1$ *bias the curve in opposite directions.*