AN INTUITIVE APPROACH TO GEOMETRIC CONTINUITY FOR PARAMETRIC CURVES AND SURFACES

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Abstract

Parametric spline curves and surfaces are typically constructed so that some number of derivatives match where the curve segments or surface patches abut. If derivatives up to order n are continuous, the segments or patches are said to meet with C^n, or n^{th} order parametric continuity. It has been shown previously that parametric continuity is sufficient, but not necessary, for geometric smoothness.

The geometric measures of unit tangent and curvature vectors for curves, and tangent plane and Dupin indicatrix for surfaces, have been used to define first and second order geometric continuity. In this paper, we extend the notion of geometric continuity to arbitrary order n (G^n) for curves and surfaces, and present an intuitive development of constraint equations that are necessary and sufficient for it. The constraints (known as the Beta constraints) result from a direct application of the univariate chain rule for curves and the bivariate chain rule for surfaces. For first and second order continuity, the Beta constraints are equivalent to requiring continuity of the geometric measures described above.

The Beta constraints provide for the introduction of quantities known as shape parameters. If two curve segments are to meet with G^n continuity, n shape parameters may be introduced. For surfaces, the use of the constraints for G^n continuity provides for the introduction of $n(n+3)$ shape functions, defined along the boundary between two surface patches. For polynomial splines, the use of the Beta constraints allows greater flexibility through the shape parameters without raising the polynomial degree.

The approach we take is important for several reasons: First, it generalizes geometric continuity to arbitrary order for both curves and surfaces. Second, it shows the fundamental connection between geometric continuity of curves and that of surfaces. Third, due to the chain rule derivation, constraints of any order can be determined more easily than using derivations based exclusively on geometric measures.

Résumé

Les courbes et surfaces paramétriques à base de splines sont généralement construites de façon à ce qu'un certain nombre de dérivées coïncident aux raccordements entre les arcs de courbe ou les carreaux de surface. Lorsqu'additionnellement les n premières dérivées sont continues, les arcs ou les carreaux se rencontrent avec continuité paramétrique C^n, ou d'ordre n. Il a déjà été établi que la continuité paramétrique est suffisante à l'obtention d'un lissage géométrique, mais qu'elle n'est pas nécessaire.

Les premier et deuxième ordres de continuité géométrique sont généralement définis à l'aide de mesures géométriques tels le vecteur tangent unitaire et le vecteur de courbure dans le cas des courbes, ainsi que le plan tangent et l'indicatrix de Dupin dans le cas des surfaces. Dans cet article, nous généralisons la no-

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*tion de continuité géométrique à n'importe quel ordre n (G^n) aussi bien pour les courbes que pour les surfaces. Nous présentons également un développement intuitif des équations de contrainte nécessaires et suffisantes. Ces contraintes, que nous appelons **les contraintes-beta**, découlent directement des règles de chaîne à une variable pour les courbes et à deux variables pour les surfaces. Pour les premier et second ordres de continuité, les contraintes-beta sont équivalentes à la continuité des mesures géométriques décrites ci-dessus.*

*Les contraintes-beta offrent l'occasion d'introduire certaines quantités connues sous le nom de **paramètres de formes**. Si deux arcs de courbe doivent se raccorder avec continuité G^n, n paramètres de formes peuvent être introduits. Pour les surfaces, l'utilisation des contraintes de continuité G^n permet d'introduire $n(n+3)$ **fonctions de formes**, lesquelles sont définies le long des limites communes entre les surfaces de deux carreaux mitoyens. Dans le cas des splines polynomiaux, l'utilisation des contraintes-beta permet une flexibilité accrue grâce aux paramètres de formes, sans pour autant augmenter le degré du polynôme.*

Cette approche est importante pour maintes raisons. Premièrement, elle généralise la notion de continuité géométriques aux ordres quelconques, autant pour les courbes que pour les surfaces. Deuxièmement, elle met en évidence la similarité fondamentale entre la continuité géométrique des courbes et celle des surfaces. Finalement, les règles de chaîne facilitent la détermination de contraintes d'ordre quelconque, comparativement à ce qu'auraient permis des dérivations basées uniquement sur des mesures géométriques.

KEYWORDS: geometric modelling, continuity, parametric curves, parametric surfaces, shape parameters.

1. Introduction

Curves are defined or *generated* by *parametrizations* (surfaces will be addressed in Section 3). A *univariate* (one variable) parametrization is a function such as $\mathbf{q}(u) = (X(u), Y(u))$, where the *domain parameter* u is allowed to range over some interval $[u_0, u_1]$. For a given value of u, the function $\mathbf{q}(u)$ can be thought of as locating a particle in Euclidean two-space. As u is increased over the interval, the particle traverses a path defined by $\mathbf{q}$, tracing out a curve in the process (see Figure 1). If $[u_0, u_1]$ is thought of as an *oriented line segment,* then $\mathbf{q}$ can be thought of as a *deformation* producing an *oriented curve*. The first derivative vector $\mathbf{q}^{(1)}$ represents the *velocity* of the particle (in general, we denote the i^{th} derivative of a univariate function by superscript (i)). The velocity is a vector quantity and, as such, contains information about orientation and *rate*, or speed. The second derivative vector $\mathbf{q}^{(2)}$ represents the *acceleration* of the particle, so it too contains information about the (change of) rate. Thus, a parametrisation contains information about the *geometry* (the shape or image of the curve), the orientation, and the rate.

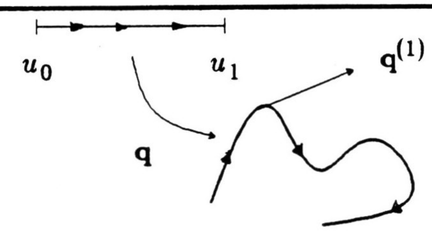

Figure 1. *The univariate parametrization* $\mathbf{q}$ *generates an oriented curve by deformation of the oriented line segment* $[u_0, u_1]$.

Figure 2 show the curves generated by three different parametrizations. The shape of the curves is identical; they differ only in orientation and rate. Curves (a) and (b) have the same orientation at each point, but the rates differ. The curve labelled (c) differs from (a) and (b) in orientation and rate. If a curve is defined to be simply the geometry property of a parametrization, one would conclude that figures (a), (b), and (c) represent *equivalent* curves. We will refer to this as the *G model* of a curve. Another possibility is to consider the geometry and orientation, which we will call the *GO model*. Using the GO model, one would say that (a) and (b) are equivalent, but (c) is different. The last possibility we will consider is the *GOR model*, where geometry, orientation, and rate are all relevant to the definition of a curve. Using this model, no pair of the curves is equivalent.

In recent years, heavy use has been made of piecewise parametric functions known as *parametric splines*. Spline curves are typically constructed by stitching together univariate parametric functions, requiring that some number of derivatives match at each *joint* (the points where the curve segments meet). If n derivatives agree at a given joint, the parametrizations there are

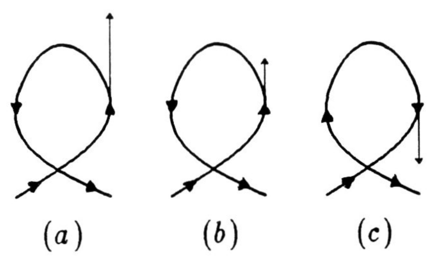

Figure 2. *Each of the curves above has the same image; they only differ in orientation and rate. Orientation is indicated by arrowheads and rate is indicated by vectors tangent to the curves.*

said to meet with n^{th} *order parametric continuity* (C^n continuity for short).

We maintain that the choice of a particular model for a curve, and hence the choice of how the curve segments are stitched together, should be application dependent. For instance, if a spline is being used to define the motion of an object in an animation system, the GOR model is most appropriate since the orientation and rate are of importance. In this type of application, parametric continuity is required to maintain the smoothness of the rate properties. In other words, parametric continuity will ensure that the object will *move* smoothly.

However, in CAGD the rate aspect of a parametrization is often unimportant. Consider for example the use of splines to describe numerically-controlled cutters. It may be necessary to specify uniquely the direction of the cutter at each point on the path, but the speed of the cutter may depend upon the hardness of the material being cut. For this type of application, the GO model is most suitable, but parametric continuity is overly restrictive since it places emphasis on irrelevant rate information. Many other applications in CAGD require only the G model, but it seems difficult to develop a useful formalism without the structure provided by orientation. We will therefore adopt the GO model, and develop an appropriate measure of continuity, one based based only on the geometry and orientation properties; we refer to this as *geometric continuity*.

It has recently come to our attention that many authors have independently defined this kind of continuity of first and second order (which we denote by G^1 and G^2, respectively) for curves and/or surfaces using geometric means. For curves, Fowler & Wilson[10], Sabin[17], Manning[13], Faux & Pratt[9], and Barsky[1] each independently defined first order continuity by requiring that the *unit tangent vectors* agree at the joints. To achieve second order continuity, both the unit tangent and *curvature* vectors were required to match. Nielson's ν-spline[14] possesses a similar kind of continuity. These geometric measures essentially ignore the rate information by "normalizing" the parametrization before determining smoothness.

For surfaces, it is common to require matching of *tangent planes* for first order geometric continuity (cf. Sabin[18] and Veron et al[20]). For surfaces of second order geometric continuity, Veron et al and Kahmann[12] require continuity of *normal curvature* in every direction, at every point on the boundary shared by the constituent surface patches. As Veron et al and Kahmann each show, this is equivalent to requiring that the *Dupin indicatrix* (cf. DoCarmo[7]) of each patch agree at the boundary curve. The Dupin indicatrix is a measure of curvature, but the curvature properties of surfaces are sufficiently complex that they cannot be characterized by something as simple as a scalar or a vector.

Although the geometric approaches described above are convenient and intuitive for first and second order continuity, a more algebraic development is better suited for the extension to continuity of higher order. The approach we take is based on *reparametrization* — the process of obtaining a new parametrization given an old one. Under the GO model, reparametrization may change rate, but not geometry or orientation. By allowing reparametrization before making a determination of continuity, the rate aspects of parametrizations may be ignored. Alternately stated, our approach is based on the following simple idea:

P1: Don't base continuity on the parametrizations at hand; reparametrize, if necessary, to obtain parametrizations that meet with parametric continuity. If this can be done, the original parametrizations must also meet smoothly, at least in a geometric sense.

The above concept is not a new one; similar principles have been discussed by Farin[8] and Veron et al[20]. What is new is the use of the principle to construct constraint equations (known as the *Beta constraints*) that are necessary and sufficient for geometric continuity of arbitrary order for both curves and surfaces. *

The Beta constraints generalise the parametric continuity constraints through the introduction of freely variable quantities called ***shape parameters.*** Once the Beta constraints are determined for a given order of continuity, they may be used in place of the parametric continuity constraints when building splines, thereby obtaining increased flexibility. For instance, if the C^2 constraints are replaced with the G^2 constraints in the uniform cubic B-spline[16], the cubic Beta-spline results[1,2]. The cubic Beta-spline is an ***approximating*** spline technique that possesses two shape parameters; an ***interpolating*** technique is described in DeRose & Barsky[6]. Faux & Pratt[9] and Farin[8] use the extra freedom allowed by geometric continuity to place ***Bézier control vertices.***

An important aspect of these techniques is that the additional flexibility of geometric continuity is added without increasing the degree of the polynomials. This is particularly important for algorithms that manipulate the spline. For instance, the complexity of Sederberg's algorithm[19] to intersect two polynomial curves of degree d grows at least as fast as d^3. Substantial savings can therefore be had by minimising the degree of the polynomials involved.

In the remainder of this paper, we extend the notion of geometric continuity to arbitrary order n (G^n) and show (in a nonrigorous way) that the derivation of the Beta constraints results from a straightforward use of the univariate chain rule for curves and the bivariate (two variable) chain rule for surfaces. For a more complete treatment, the reader is referred to Barsky & DeRose[3] and DeRose[5].

2. Geometric Continuity for Curves

We begin the study of geometric continuity for curves by examining the reparametrisation process. Two parametrizations are said to be ***GO-equivalent*** if they have the same geometry and orientation in the neighborhood of each point. Given a parametrisation $\mathbf{q}$, all GO-equivalent parametrisations may be obtained by ***functional composition.*** More specifically, if $\mathbf{q}(u)$ and $\tilde{\mathbf{q}}(\tilde{u})$ are GO-equivalent, then they are related by $\tilde{\mathbf{q}}(\tilde{u}) = \mathbf{q}(u(\tilde{u}))$, for some appropriately chosen ***change of parameter*** $u(\tilde{u})$ (see Figure 3). Since $\mathbf{q}$ and $\tilde{\mathbf{q}}$ must have the same orientation, u must be an increasing function of $\tilde{u}$, implying that u must satisfy the ***orientation preserving condition*** $u^{(1)} > 0$. Intuitively, $u(\tilde{u})$ deforms the interval $[\tilde{u}_0, \tilde{u}_1]$ into the interval $[u_0, u_1]$ without reversing the orientation of the segment $[\tilde{u}_0, \tilde{u}_1]$. This in turn implies that $\mathbf{q}$ and $\tilde{\mathbf{q}}$ will have the same geometry and orientation, but they may differ in rate.

* Goodman[11] and Ramshaw[15] have independently derived the univariate Beta constraints from the univariate chain rule.

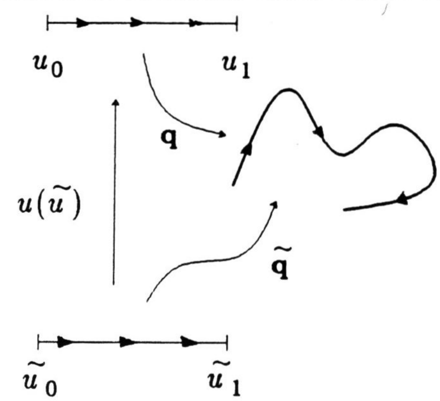

Figure 3. *The equivalent parametrizations* $\mathbf{q}$ *and* $\tilde{\mathbf{q}}$ *are related by the change of parameter* $u(\tilde{u})$.

A univariate parametrisation is ***regular*** if the first derivative vector does not vanish. It is well known from differential geometry[7] that regularity is, in general, essential for the smoothness of the resulting curve. We will therefore restrict the discussion to regular parametrizations. We now give a more precise definition of G^n continuity:

Definition 1: *Let* $\mathbf{r}(t), t \in [t_0, t_1]$ and $\mathbf{q}(u), u \in [u_0, u_1]$ *be two parametrisations such that* $\mathbf{r}(t_1) = \mathbf{q}(u_0)$ *(see Figure 4). These parametrisations meet with* G^n *continuity at J if and only if there exist GO-equivalent parametrisations* $\tilde{\mathbf{r}}(\tilde{t})$ *and* $\tilde{\mathbf{q}}(\tilde{u})$ *that meet with* C^n *continuity.*

Definition 1 is simply a restatement of principle P1, but in practice one cannot examine all GO-equivalent parametrisations in an effort to find two that meet with parametric continuity. However, it is possible to find conditions on $\mathbf{r}$ and $\mathbf{q}$ that are necessary and sufficient for the ***existence*** of GO-equivalent parametrisations that meet with parametric continuity.

Although Definition 1 suggests that both $\mathbf{r}$ and $\mathbf{q}$ need to be reparametrized, it is possible to show that

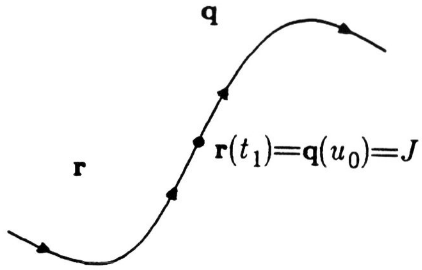

Figure 4. *The parametrizations* $\mathbf{r}(t)$ *and* $\mathbf{q}(u)$ *meet at the common point* $\mathbf{J}$.

Definition 1 holds if and only if there exists a $\tilde{\mathbf{q}}$ that meets $\mathbf{r}$ with parametric continuity. In other words, only one of the parametrisations needs to be reparametrised to determine smoothness.

We will ultimately be interested in the derivative properties of $\tilde{\mathbf{q}}$. The *univariate chain rule* allows us to express derivatives of $\tilde{\mathbf{q}}$ in terms of the derivatives of $\mathbf{q}$ and u. For example, the first derivative is given by

$$\begin{aligned}\tilde{\mathbf{q}}^{(1)} = \frac{d\tilde{\mathbf{q}}}{d\tilde{u}} &= \frac{d\mathbf{q}(u(\tilde{u}))}{d\tilde{u}} \\ &= \frac{du}{d\tilde{u}}\frac{d\mathbf{q}}{du} \\ &= u^{(1)}\,\mathbf{q}^{(1)}.\end{aligned} \tag{2.1}$$

In general, the i^{th} derivative of $\tilde{\mathbf{q}}$ can be written as some function, call it $\mathcal{CR}_i$, of the first i derivatives of u and $\mathbf{q}$. That is,

$$\tilde{\mathbf{q}}^{(i)} = \mathcal{CR}_i(\mathbf{q}^{(1)},\cdots,\mathbf{q}^{(i)}, u^{(1)},\cdots,u^{(i)}). \tag{2.2}$$

We are actually interested in $\tilde{\mathbf{q}}^{(i)}$ evaluated at its *left parametric endpoint* $\tilde{u}_0$. Thus, derivatives of $\mathbf{q}$ and u must also be evaluated at their left endpoints:

$$\tilde{\mathbf{q}}^{(i)}(\tilde{u}_0) = \mathcal{CR}_i(\mathbf{q}^{(1)}(u_0),\cdots,\mathbf{q}^{(i)}(u_0), u^{(1)}(\tilde{u}_0),\cdots,u^{(i)}(\tilde{u}_0)). \tag{2.3}$$

Since u is a scalar function, evaluating one of its derivatives results in a real number. In particular, let $u^{(j)}(\tilde{u}_0) = \beta_j$, $j = 1,...,i$. Equation (2.3) then becomes

$$\tilde{\mathbf{q}}^{(i)}(\tilde{u}_0) = \mathcal{CR}_i(\mathbf{q}^{(1)}(u_0),\cdots,\mathbf{q}^{(i)}(u_0), \beta_1,\cdots,\beta_i). \tag{2.4}$$

The orientation preserving quality of u implies that $\beta_1 > 0$.

We are now in a position to state the primary result of geometric continuity for curves. Recall that $\mathbf{r}$ and $\mathbf{q}$ meet with G^n continuity if $\mathbf{q}$ can be reparametrised to $\tilde{\mathbf{q}}$ so that derivatives of $\mathbf{r}$ and $\tilde{\mathbf{q}}$ agree. That is, we require that

$$\mathbf{r}^{(i)}(t_1) = \tilde{\mathbf{q}}^{(i)}(\tilde{u}_0), \qquad i = 1,...,n. \tag{2.5}$$

Positional continuity is implicitly assumed (see Figure 4). Substituting equation (2.4) into (2.5) yields

$$\mathbf{r}^{(i)}(t_1) = \mathcal{CR}_i(\mathbf{q}^{(1)}(u_0),\cdots,\mathbf{q}^{(i)}(u_0), \beta_1,\cdots,\beta_i) \qquad i = 1,...,n. \tag{2.6}$$

The constraints resulting from equation (2.6) are the *univariate Beta constraints* and the numbers $\beta_1,...,\beta_n$ are the *shape parameters*. The above discussion is not a proof that the Beta constraints are necessary and sufficient conditions for geometric continuity, but such a proof can be constructed[3,5]. Thus, if equations (2.6) are satisfied for any choice of the βs, subject to $\beta_1 > 0$, then the coincident curve segments will meet with G^n continuity. For instance, the Beta constraints for G^4 continuity between $\mathbf{r}$ and $\mathbf{q}$ are

$$\begin{aligned}\mathbf{r}^{(1)}(t_1) &= \beta_1\,\mathbf{q}^{(1)}(u_0) \\ \mathbf{r}^{(2)}(t_1) &= \beta_1^2\,\mathbf{q}^{(2)}(u_0) + \beta_2\,\mathbf{q}^{(1)}(u_0) \\ \mathbf{r}^{(3)}(t_1) &= \beta_1^3\,\mathbf{q}^{(3)}(u_0) + 3\beta_1\beta_2\,\mathbf{q}^{(2)}(u_0) \\ &\quad + \beta_3\,\mathbf{q}^{(1)}(u_0) \\ \mathbf{r}^{(4)}(t_1) &= \beta_1^4\,\mathbf{q}^{(4)}(u_0) + 6\beta_1^2\beta_2\,\mathbf{q}^{(3)}(u_0) \\ &\quad + (4\beta_1\beta_3 + 3\beta_2^2)\,\mathbf{q}^{(2)}(u_0) + \beta_4\,\mathbf{q}^{(1)}(u_0).\end{aligned} \tag{2.7}$$

Although equations (2.7) were derived using the chain rule, the first two are identical to the constraints resulting from a geometric derivation of unit tangent and curvature vector continuity[2,13]. Thus, our approach reduces to previous definitions of G^1 and G^2 continuity for curves. It can also be shown that Beta constraints for n^{th} order continuity are equivalent to requiring continuity of the first n derivatives with respect to *arc length* [3,5].

When constructing a spline technique, if the Beta constraints are used in place of the parametric continuity constraints, new freedom is introduced through the shape parameters. These parameters may be made available to a designer in a CAGD environment to change the shape of the target curve, as the following example shows.

Example 2.1: To demonstrate the use of the Beta constraints, we will sketch the construction of the geometric continuous analogue of the uniform quartic B-spline called (naturally enough) the quartic Beta-spline.

The j^{th} segment of the quartic B-spline is generated by

$$\mathbf{q}_j(u) = \sum_{k=-2}^{2} \mathbf{V}_{j+k} B_k(u), \qquad u \in [0,1] \tag{2.8}$$

where the *basis functions* $B_k(u)$ are quartic polynomials that satisfy

$$B_{k+1}^{(i)}(1) = B_k^{(i)}(0), \quad i = 0,1,2,3, \quad k = -2,\ldots,1. \tag{2.9}$$

The sequence of *control vertices* $\mathbf{V}_{j+k}$ comprise a *control polygon*.

Since the derivative properties of the basis functions are inherited by $\mathbf{q}_j$, equation (2.9) implies that the curve segments meet with C^3 continuity. The quartic Beta-spline is constructed by building quartic polynomials $b_k(u)$ that satisfy the G^3 constraints instead of the C^3 constraints of equation (2.9). That is,

$$b_{k+1}^{(i)}(1) = \mathcal{CR}_i(b_k^{(1)}(0),\cdots,b_k^{(i)}(0), \beta_1,\cdots,\beta_i) \quad i = 0,1,2,3. \tag{2.10}$$

Equation (2.10) implies that the basis functions are dependent upon the shape parameter values. Changing a shape parameter therefore changes the shape of the resulting curve (see Figure 5). •

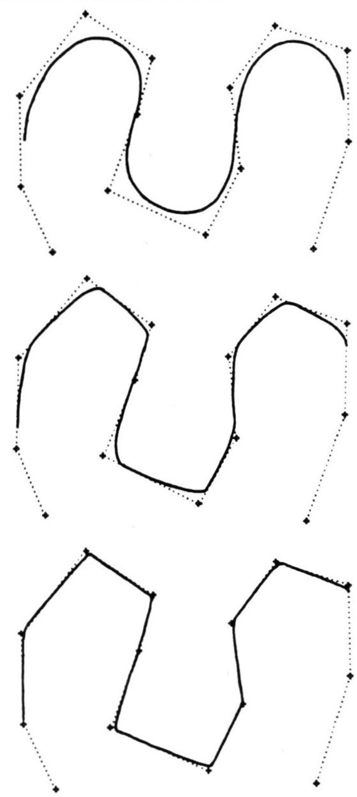

Figure 5. *The curves above share the same control polygon, and all have $\beta_1 = 1$ and $\beta_3 = 0$; they differ only in the value of β_2. The top curve has $\beta_2 = 0$, the middle curve has $\beta_2 = 20$, and the bottom curve has $\beta_2 = 100$.*

3. Geometric Continuity for Surfaces

In this section, we extend the notions of geometric continuity to surfaces. Since care was taken in Section 2 not to base the development of geometric continuity on concepts (such as arc length) that don't apply to surfaces, the machinery developed for univariate parametrisations can be readily extended to bivariate parametrisations.

A *surface patch* is defined by a bivariate function such as $\mathbf{G}(u,v) = (X(u,v), Y(u,v), Z(u,v))$, where u and v are allowed to range over some region D of the uv plane (see Figure 6). Loosely speaking, a *surface* is a collection of surface patches. We use the notation $\mathbf{G}^{(i,j)}(u,v)$ to denote the i^{th} partial derivative with respect to u, and the j^{th} partial with respect to v. In general, a superscript (i,j) denotes the i^{th} partial with respect to the first variable, and the j^{th} partial with respect to the second. A bivariate parametrisation such as $\mathbf{G}$ is *regular* if the first order partials ($\mathbf{G}^{(1,0)}$ and $\mathbf{G}^{(0,1)}$) are linearly independent; we will deal exclusively with regular parametrisations.

In Section 1, we saw that univariate parametrisations contain information about geometry, orientation, and rate. The same is true of bivariate parametrisations. Orientation can be defined by treating D as an *oriented plane* having a "top side" and a "bottom side." $\mathbf{G}$ can then be thought of as deforming the oriented plane to produce an oriented, or two-sided, surface patch. The rate information enters through the partial derivatives of the parametrisation. We can therefore speak of the G, GO, and GOR models of surfaces. Just as for curves,

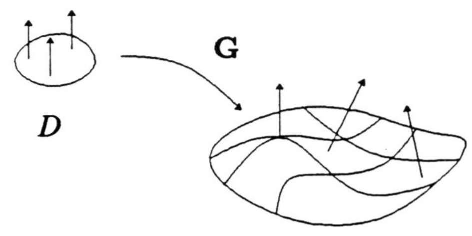

Figure 6. *The bivariate parametrization* **G** *deforms the oriented domain D to generate an oriented surface patch.*

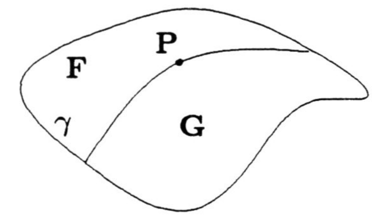

Figure 7. *The surface patches generated by the parametrizations* **F** *and* **G** *meet at the boundary curve* γ.

the use of a particular model should be application dependent. We will adopt the GO model for two reasons: first, orientation is necessary in applications, such as rendering, where the two-sidedness of surfaces is important, and second, it seems difficult to develop a useful formalism without the structure provided by orientation, especially when surfaces are allowed to intersect themselves.

We now examine the reparametrisation process for surface patches. Two bivariate parametrisations are GO-equivalent if they have the same geometry and orientation in the neighborhood of each point on the surface patch. If $\mathbf{G}(u,v)$ and $\widetilde{\mathbf{G}}(\tilde{u},\tilde{v})$ are GO-equivalent, then they are related by

$$\widetilde{\mathbf{G}}(\tilde{u},\tilde{v}) = \mathbf{G}(u(\tilde{u},\tilde{v}), v(\tilde{u},\tilde{v})) \qquad (3.1)$$

where the functions u and v satisfy the orientation preserving condition **

$$u^{(1,0)}v^{(0,1)} - u^{(0,1)}v^{(1,0)} > 0. \qquad (3.2)$$

We now examine how surface patches are stitched together with parametric continuity. Referring to Figure 7, $\mathbf{F}(s,t)$ and $\mathbf{G}(u,v)$ meet with n^{th} order parametric continuity if and only if all like partial derivatives of order up to n agree for each point of the boundary curve. That is,

$$\mathbf{F}^{(i,j)}(\gamma) = \mathbf{G}^{(i,j)}(\gamma), \qquad i+j = 1,\ldots,n, \qquad (3.3)$$

where evaluation at γ is to be interpreted as evaluation at all points **P** of γ.

** Readers familiar with multivariate calculus may recognize equation (3.2) as the Jacobian of the change of parametrization (cf. DoCarmo[7]).

Just as for curves, parametric continuity is appropriate for the GOR model of a surface, but it is not suitable for use with the GO model since it places emphasis on irrelevant rate information. The determination of continuity can be made insensitive to rate by allowing reparametrization. Thus, we say that **F** and **G** meet with G^n continuity if and only if there exist GO-equivalent parametrizations $\widetilde{\mathbf{F}}$ and $\widetilde{\mathbf{G}}$ that meet with C^n continuity.

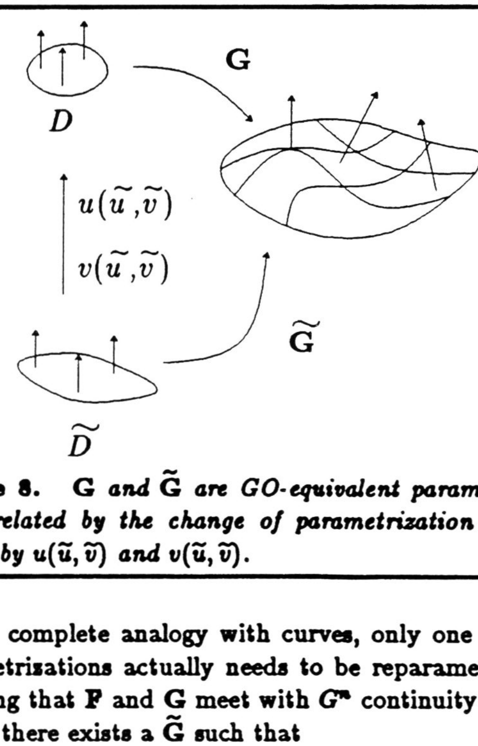

Figure 8. **G** *and* $\widetilde{\mathbf{G}}$ *are GO-equivalent parametrizations related by the change of parametrization determined by* $u(\tilde{u},\tilde{v})$ *and* $v(\tilde{u},\tilde{v})$.

In complete analogy with curves, only one of the parametrisations actually needs to be reparametrised, implying that **F** and **G** meet with G^n continuity if and only if there exists a $\widetilde{\mathbf{G}}$ such that

$$\mathbf{F}^{(i,j)}(\gamma) = \widetilde{\mathbf{G}}^{(i,j)}(\gamma), \qquad i+j = 1,\ldots,n. \qquad (3.4)$$

Once again, in complete analogy with curves, the bivariate chain rule can be used to express derivatives of $\widetilde{\mathbf{G}}$

in terms of $\mathbf{G}$. In general, the i,j^{th} partial of $\tilde{\mathbf{G}}$ can be expressed as some function, call it $\mathcal{CR}_{i,j}$, of the partials of $\mathbf{G}$, u, and v, up to order $i+j$. Stated mathematically,

$$\tilde{\mathbf{G}}^{(i,j)} = \mathcal{CR}_{i,j}(\mathbf{G}^{(k,l)}, s^{(k,l)}, t^{(k,l)}), \tag{3.5}$$

where the indices (k,l) are to take on all positive values such that $k+l=i+j$.

We may now obtain the *bivariate Beta constraints* by evaluating (3.5) along the boundary curve, followed by substitution into equation (3.4) to get

$$\mathbf{F}^{(i,j)}(\gamma) = \mathcal{CR}_{i,j}(\mathbf{G}^{(k,l)}(\gamma), s^{(k,l)}(\gamma), t^{(k,l)}(\gamma)) \tag{3.6}$$

for $k+l=i+j$ and $i+j=1,...,n$.

The equations resulting from (3.6) are the bivariate Beta constraints, and the scalar functions $u^{(k,l)}(\gamma)$ and $v^{(k,l)}(\gamma)$ are the shape functions. A simple counting argument shows that it is possible to introduce $n(n+3)$ shape functions when two patches are stitched together with G^n continuity.

Just as the univariate Beta constraints can supplant the parametric continuity constraints when building spline curves, the bivariate Beta constraints can replace the parametric constraints when building spline surfaces. It can be shown that the Beta constraints for first and second order are equivalent to requiring continuity of tangent planes and Dupin incatrices of the patches match along the boundary curve[5]. Thus, the chain rule approach agrees with geometric intuition for both G^1 and G^2 continuity. Moreover, the chain rule approach yields the second order constraints with less effort than the geometric approach. For higher order continuity, geometric intuition becomes more feeble, but the chain rule approach still applies.

4. Conclusion

We have defined n^{th} order geometric continuity for parametric curves and surfaces, and derived the Beta constraints that are necessary and sufficient for it. The derivation of the Beta constraints is based on a simple principle of reparametrization in conjunction with the univariate chain rule for curves, and the bivariate chain rule for surfaces. This approach therefore uncovers the connection between geometric continuity for curves and geometric continuity for surfaces, provides new insight into the nature of geometric continuity in general, and allows the determination of the Beta constraints with less effort than previously required.

The use of the Beta constraints for G^n continuity allows the introduction of n shape parameters for curves, and $n(n+3)$ shape functions for surfaces. The shape parameters and shape functions may be used to modify the shape of a geometrically continuous curve or surface, respectively. However, geometric continuity is only appropriate for applications where rate aspects of the parametrizations are unimportant since discontinuities in rate are allowed.

As a final comment, the approach we have taken is not based on measures that are inherent to curves and surfaces, so the generalization to k-variate objects (volumes, hyper-volumes, etc.) can be made very simply: two k-variate parametrizations are GO-equivalent if they are related by a change of parametrization with positive Jacobian; the corresponding Beta constraints may be derived in complete analogy to the development of Section 3, using the k-variate chain rule[4] in place of the bivariate chain rule.

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