Fitting Triangular B-Splines to Functional Scattered Data

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Abstract

Scattered data is, by definition, irregularly spaced. Uniform surface schemes are not well adapted to the locally varying nature of such data. Conversely, Triangular B-Spline surfaces [DMS92] are more flexible in that they can be built over arbitrary triangulations and thus can be adapted to the scattered data. This paper discusses the use of DMS spline surfaces for approximation of scattered data. A method is provided for automatically triangulating the domain containing the points and generating basis functions over this triangulation. A surface approximating the data is then found by a combination of least squares and bending energy minimization. This combination serves both to generate a smooth surface and to accommodate for gaps in the data. Examples are presented which demonstrate the effectiveness of the technique for mathematical, geographical and other data sets.

Keywords: Scattered Data Approximation, Triangular B-Splines, DMS Splines, Simplex Splines, Quadtrees

1 Introduction

Scattered data fitting problems come from many different areas, including scientific and engineering data visualization, mining and mapping, geographic information systems, meteorology and other fields [Nie93]. Often we would like to visualize the geometry of the surface that corresponds to a set of scattered data in order to better understand the underlying data.

The problem we address is the fitting of a functional surface $F(x, y)$ to a collection of scattered functional data $\{(x_i, y_i, z_i(x_i, y_i))\}$. Our goal is to find a smooth surface F that is a reasonable approximation to the data.

The kind of functional surface we choose is the DMS spline formulation ([DMS92]). The DMS spline surfaces have numerous positive characteristics that make them appropriate for this data fitting problem, including their automatic smoothness properties, the ability to define a surface over an arbitrary triangulation (which can be adapted to the local density of sampled data) and their "completeness," in that all piecewise polynomials of a particular degree over a given triangulation can be represented by a DMS spline of the same degree.

Schumaker introduces the problem of scattered data approximation in [Sch76]. Auerbach *et al.* [AMNS91] examine the functional data approximation problem for the simplex spline space described in [Höl82, DM82]. We find that they do not fully address the problem of producing a good triangulation of a data set, nor do they describe how to deal with data sets that cause difficulties when using the least squares method. In [BHS93], Brunnett *et al.* discuss the use of combined least squares and fairing functional minimization in the design of quintic tensor product B-splines surfaces to achieve C^1-continuous surfaces. They do not address the problem of partitioning the domain in a manner appropriate to a given set of data, in this case the definition of parametric knot lines.

The following describes the organization of the paper. In Section 2 we review the definition of simplex splines and that of the DMS spline scheme. In Section 3, we examine how a triangulation of the domain may be formed that adapts to the distribution of sample data. In Section 4, we look at the minimization problem used to find the approximating surface, namely the *least squares method* and describe how it can be augmented with a smoothing term in order to overcome certain difficulties. Lastly, we present our conclusions and suggestions for further work in Section 5. Examples fitting quadratic DMS splines (with C^1 continuity) to various data sets appear as figures throughout the paper.

2 DMS splines

A DMS spline surface is formed as a linear combination of basis functions. These basis functions are defined over a triangulation of $\mathbb{R}^2$. This section explains how these basis functions are constructed. An introduction to this material can be found in [Sei91].

Bivariate simplex splines form the individual basis functions for the DMS spline surface [Mic79]. Simplex splines are defined *via* affine combinations of points. Formally, consider a point u and a set of points $V = \{t_0, \ldots, t_{n+2}\}$, called *knots*, from $\mathbb{R}^2$. The degree n simplex spline $M(u|V)$ is defined recursively as follows:

• For $n > 0$, with $V = \{t_0, \ldots, t_{n+2}\}$, select three points $W = \{t_{i_0}, t_{i_1}, t_{i_2}\}$ from V, such that W is affinely independent. Then

$$M(u|V) = \sum_{j=0}^{2} \lambda_j(u|W)\, M(u|V \backslash \{t_{i_j}\}) \qquad (1)$$

where $u = \sum_{j=0}^{2} \lambda_j(u|W) t_{i_j}$ and $\sum_{j=0}^{2} \lambda_j(u|W) = 1$.

• For $n = 0$, with $V = \{t_0, t_1, t_2\}$,

$$M(u|t_0, t_1, t_2) = \frac{\chi_{[t_0,t_1,t_2)}(u)}{2|\Delta(t_0, t_1, t_2)|}, \qquad (2)$$

$$\text{where } \chi_{[t_0,t_1,t_2)} = \begin{cases} 1 & \text{if } u \in [t_0, t_1, t_2) \\ 0 & \text{otherwise} \end{cases}$$

with $[t_0, t_1, t_2)$ being the *half-open convex hull* [1] of points t_0, t_1 and t_2

A slightly modified version of Equation 1 can be used to find the derivative of $M(u|V)$ with respect to a parametric vector v:

$$D_v\, M(u|V) = n \sum_{j=0}^{2} \mu_j(v|W) M(u|V \backslash \{t_{i_j}\}) \qquad (3)$$

with $v = \sum_{j=0}^{2} \mu_j(v|W) t_{i_j}$ and $\sum_{j=0}^{2} \mu_j(v|W) = 0$.

Simplex splines possess a number of interesting properties. They are piecewise polynomial of degree n, are zero outside the convex hull of the knots V and non-negative within it, and are smooth, in the sense that if the knots of V are in general position, then $M(u|V)$ exhibits C^{n-1} continuity.

Simplex splines can now be combined to form a spline surface [DMS92]. Let T be an arbitrary proper triangulation of $\mathbb{R}^2$ or some bounded domain $D \subset \mathbb{R}^2$. "Proper" means that every pair of domain triangles I, J are disjoint, or share exactly one edge, or exactly one vertex.

To each vertex t_i of the triangulation T, we assign a *knot cloud*, which is a sequence of points (*knots*) $t_{i,0}, \ldots, t_{i,n}$, where $t_{i,0} \equiv t_i$. For each triangle $\Delta = (t_0, t_1, t_2) \in T$, we require that $(t_{0,i}, t_{1,j}, t_{2,k})$ always form a proper triangle. We then define, for each Δ and $i + j + k = n$, the knot sets

$$V_{i,j,k}^{\Delta} = \{t_{0,0}, \ldots, t_{0,i}, t_{1,0}, \ldots, t_{1,j}, t_{2,0}, \ldots, t_{2,k}\} \quad (4)$$

which yields $\binom{n+2}{n}$ simplex splines $M(u|V_{i,j,k}^{\Delta})$.

The *normalized B-splines* are then defined as $N_{i,j,k}^{\Delta}(u) = d_{i,j,k}^{\Delta} M(u|V_{i,j,k}^{\Delta})$, where $d_{i,j,k}^{\Delta}$ is defined to be twice the area of $\Delta(t_{0_i}, t_{1_j}, t_{2_k})$. This normalization ensures that the basis functions sum to one.

A functional surface F of degree n over the triangulation T with *knot net* $\mathcal{K} = \{t_{i,l} | i \in \mathbb{Z}, l = 0, \ldots, n\}$ is then defined as

$$F(u) = \sum_{\Delta \in T} \sum_{i+j+k=n} c_{i,j,k}^{\Delta} N_{i,j,k}^{\Delta}(u), \qquad (5)$$

with coefficients $c_{i,j,k}^{\Delta} \in \mathbb{R}$.

These normalized B-spline surfaces demonstrate properties similar to B-spline curves, in that they are affine invariant, have local control, and exhibit the Convex Hull Property. Moreover, every degree n piecewise polynomial over a triangulation T can be represented as a normalized B-spline surface.

3 Finding a triangulation

DMS spline basis functions are defined with respect to a triangulation of the domain, so our first task is to generate an appropriate triangulation of the portion of the domain containing our sampled data. Once a set of basis functions has been determined, an approximating surface F in the span of these functions can be found by functional minimization, as described in Section 4.

3.1 Properties of a good triangulation

The triangulation we form should possess the following four properties: *Property One*, all sample points

[1] u is in $[t_0, t_1, t_2)$ if the set $\{u + s\eta + t\xi \mid s, t > 0,\ s + t < \epsilon\}$ is contained within the convex hull of $\{t_0, t_1, t_2\}$, for some $\epsilon > 0$, ξ being the horizontal unit vector in $\mathbb{R}^2$ and η a vector with positive slope [Sei91].

must be contained in some triangle of the triangulation. *Property Two,* no triangle has too many or too few sample points within it, and points within each triangle are distributed as uniformly as possible. *Property Three,* triangles are not too elongated. *Property Four,* neighbouring triangles are roughly comparable in size.

Property One ensures that all of the data has an influence on the final surface. Property Two ensures that each data point is well-represented by the surface. Properties Three and Four ensure that we get a "good" set of basis functions. Very thin triangles could cause numerical problems in evaluation.

Unfortunately, in extreme cases, our goals may not be consistent with one another. For example, if all our sample points are nearly collinear, then to satisfy Property Two we should probably have elongated triangles in our triangulation. This would violate Property Three.

It is clear that some other means of triangulating the data points must be used that adapts the size of triangles used to the local density of the data. The initial temptation is to use the sample (x, y) values as vertices of a Delaunay triangulation [For94]. This is, of course, ridiculous, as this choice results in an explosion of triangles. Another adaptive data structure commonly used for 2D domains is the quadtree, which we will use in the next section in order to build a triangulation.

3.2 Quadtree division of the domain

We now describe a triangulation scheme that satisfies Properties One, Three and Four and partially satisfies Property Two. We will rely on the minimization technique of Section 4.2 to ameliorate the remaining difficulties with the triangulation scheme.

We begin by finding a bounding box around the data, and generate a quadtree partition of this box such that no leaf node contains more than a specified number of sample points. The corners of leaf nodes generated by the quadtree division can be used as vertices of (relatively) equally sized triangles.

This simple quadtree violates Property Three for data sets that contain tight clusters of points. Point clusters can cause finely-subdivided areas to be adjacent to far more coarsely subdivided areas, which, in turn, generates long thin triangles. This can be remedied by requiring that the quadtree be *balanced* [Sam90]. A quadtree is balanced if the depths of two adjacent leaf nodes differ by at most one.

A potential problem exists when the bounding box is a poor fit to the data, that is, when large areas within the bounding box contain no data. In this case, one could accept user input to decide whether to exclude leaf nodes containing no sample points from the triangulation.

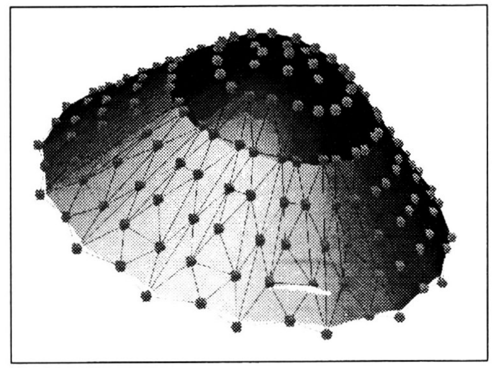

Figure 1: Capping a truncated cone by minimizing thin-plate energy

3.3 Assigning knot clouds

Once a triangulation has been formed, knot clouds are added to vertices of the triangulation. The knot clouds are chosen so as to avoid collinearity of knots associated with a particular triangle. The method used is similar to the one given in [AMNS91, page 81]. The selection of the knot clouds then defines the basis functions for the surface.

Knots can also be selected to promote collinearity along certain edges of the triangulation. If $k + 2$ of the knots used to define a particular DMS basis function are placed collinearly, then the continuity of the surface along that parametric line will be reduced by k. Reduced continuity can be introduced into the surface using this method.

Once a triangulation and its knot clouds have been defined, linear combinations of the corresponding DMS basis functions can be used to form surfaces. In the next section we discuss the problem of selecting, from among this set of surfaces, some surface F that is a good approximation to the data.

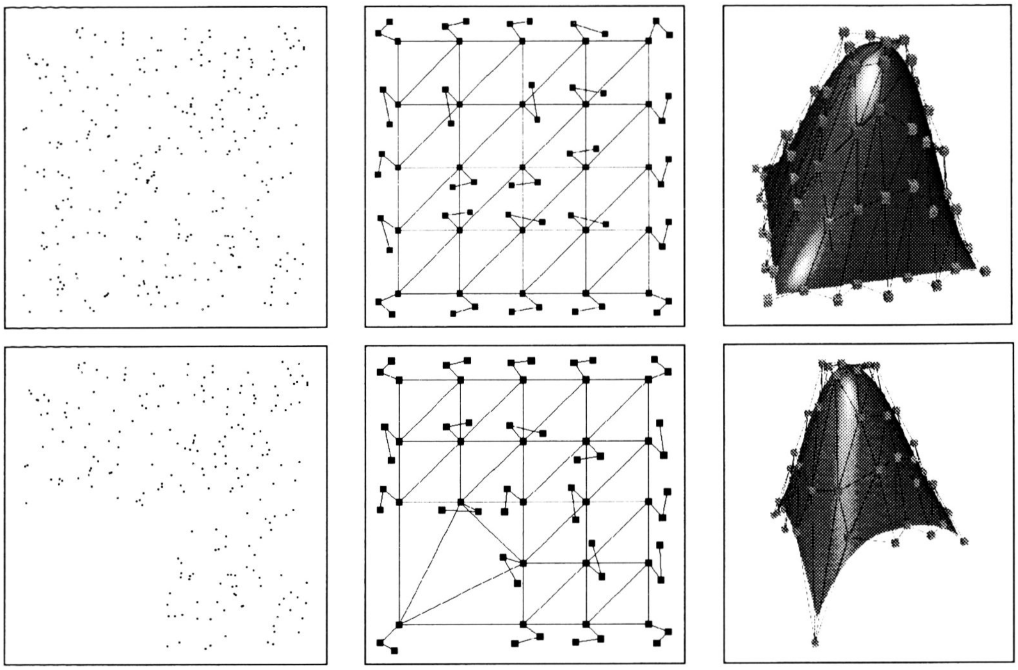

Figure 2: Top Row: *Left,* the locations in $[0,1] \times [0,1]$ of data points where the function $\sin(\pi x)\sin(\pi y)$ is randomly sampled. *Middle,* quadtree-based triangulation of the region, showing the knot cloud associated with each vertex. *Right,* surface fit using pure least squares, with control points. Bottom Row: *Left,* as above, but with data missing from the region $[0,0.5] \times [0,0.5]$. *Middle,* corresponding triangulation with knot clouds. Triangles in the lower left corner contain no data. *Right,* surface fit using a combination of least squares and thin plate energy minimization. The front corner of the surface corresponds to the domain triangles that have no data. Because of the missing data, this data set cannot be fit without the assistance of a smoothing term.

4 Least squares fitting and fairing by minimizing a functional

In this section, we assume that a fixed set of basis functions exists, and that a surface F in the span of those functions should be used to approximate the data.

4.1 Least squares minimization

We use the least squares method (see [HL89, Die93]) for approximating a set of functional scattered data. Given a surface $F(x,y)$, the unweighted least squares functional $LS(F) = \sum_l (F(x_l,y_l) - z_l)^2$ provides a measure of how well F approximates the data. If we minimize this sum, we will obtain a good approximation to the data. Since F is the linear combination of some fixed set of basis functions, the functional LS is quadratic in their coefficients, and so the minimization problem can be expressed as a linear system in the coefficients of those basis functions. Standard solution techniques form either the *observation equations* or *normal equations* for a given data set [Die93]. Both kinds of systems can be solved using standard techniques.

In order to form the individual entries of either of these matrices, each basis function (in our case, each DMS basis function $N^{\Delta}_{i,j,k}$) must be evaluated at each (x_l, y_l) location. This poses no particular difficulty, as evaluation algorithms for DMS splines based on Equation 1 are readily available [PS94, FS93].

Formulating the approximation problem as a least squares problem has a number of advantages and disadvantages. The main advantages of the least squares method are that it is simple to understand and relatively easy to implement. The chief disadvantage is that the solution is very sensitive to the

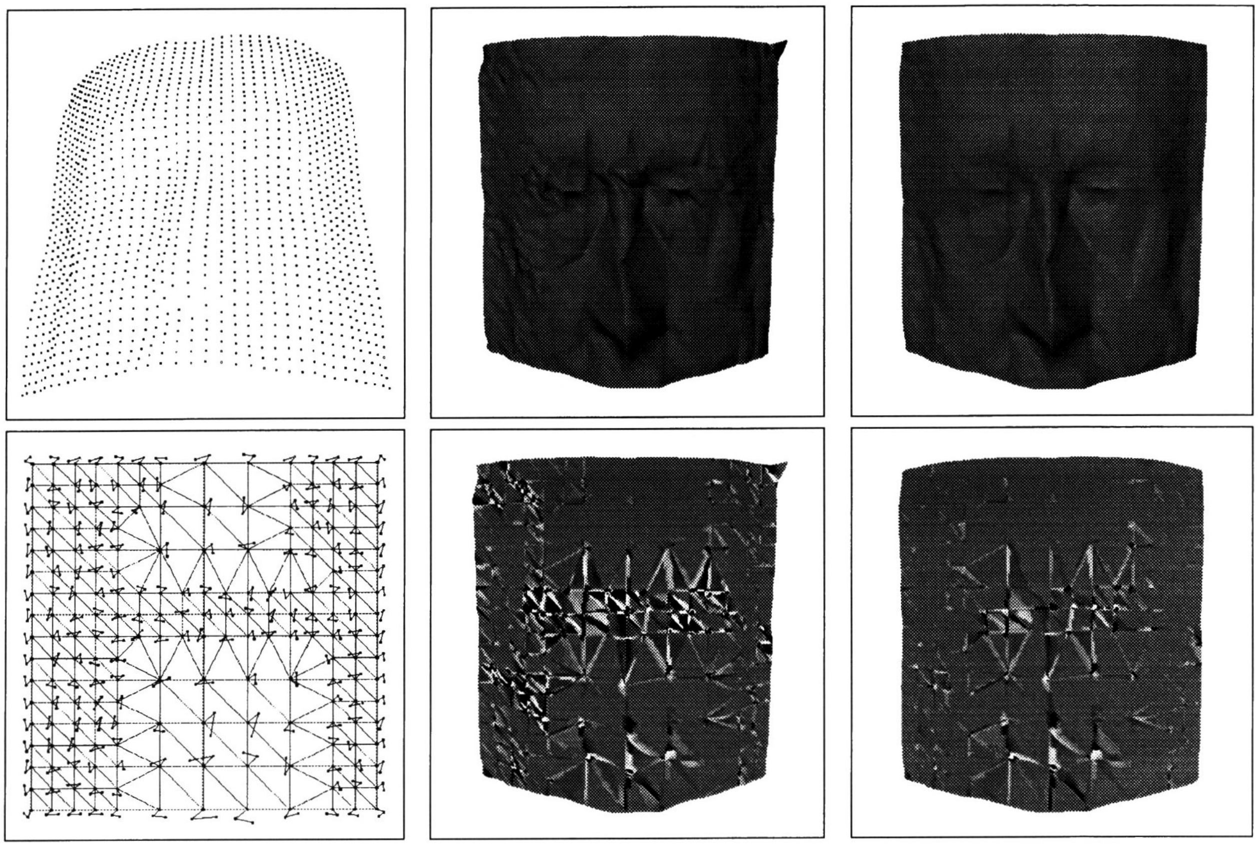

Figure 3: Fitting a surface to data taken from a bust of Victor Hugo (data courtesy the University of Waterloo Computer Graphics Lab). *Top Left,* the data points in 3D. *Bottom Left,* corresponding triangulation with knot clouds. *Top Middle,* surface fit using pure least squares. *Top Right,* surface fit using mixed least squares and thin plate energy minimization ($\alpha = 0.25$) *Bottom Middle/Right,* corresponding plots of Gaussian curvature. Darker regions have negative curvature, lighter regions have positive curvature. The mixed fit surface exhibits far less variation in curvature.

location of data points with respect to the given set of basis functions. A particular basis function may have no influence on the value of the least squares functional (because it contains no data points within its region of support), or a number of basis functions may be linearly dependent with respect to the given data. Moreover, the surface determined by the least squares approach may very well lie close to data points, but may not be very smooth.

In order to deal with these problems we modify the least squares functional to take surface smoothness into account. The smoothness factor can be used to assign reasonable values to coefficients undetermined by the data, by choosing values that make the surface as smooth as possible.

4.2 Smoothing by minimizing thin plate energy

"Fairing" is the process of reducing irregularities in a surface in order to make it smoother. The typical fairing process proceeds by defining a fairness functional for a given set of surfaces, and finding the minimum surface F with respect to that functional. Different functionals have been defined and used for this purpose [Gre94a, Gre94b].

If we choose a functional that is quadratic in the basis function coefficients, then as is the case with the least squares functional, it is possible to express the minimization problem as a linear system that can be solved using matrix techniques.

Since we are dealing with *functional* DMS sur-

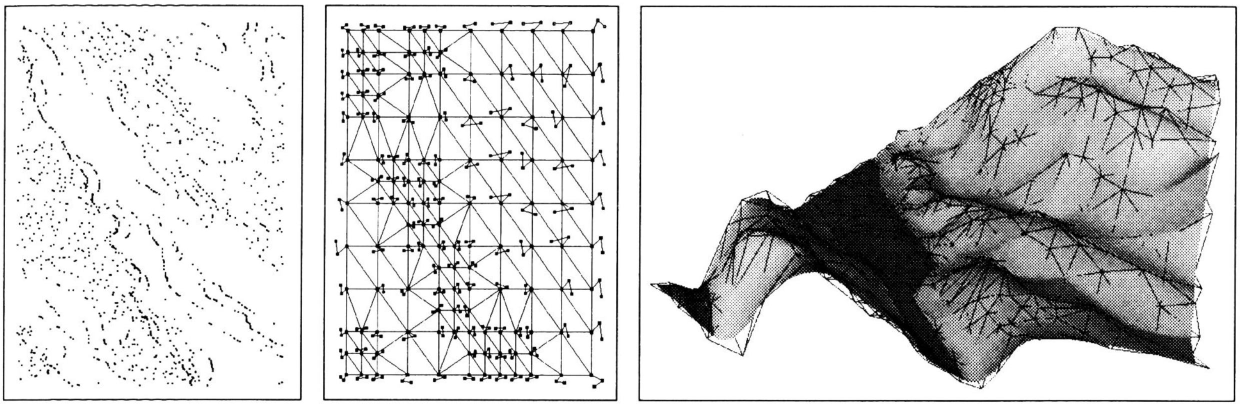

Figure 4: Geographic elevation data for a portion of Île D'Orléans, downstream from Québec City (US Geological Survey data set *quebec-e* [EC85]). *Left,* data sampled at 21m contour levels (East is to the top). *Middle,* the quadtree-based triangulation with knot clouds. *Right,* side view of the surface fitted using a combination of least squares and thin plate energy minimization, with control net (surface patches lying substantially in the St. Lawrence River are darkly coloured). The vertical scale has been exaggerated 40 times to enhance detail.

faces, an appropriate smoothing functional is the *linearized thin plate energy* functional $J(F(x,y))$ [Gre94a, Gre94b, CG91, HKD93], defined as

$$J(F) = \int_\Omega F_{xx}^2 + 2F_{xy}^2 + F_{yy}^2 \, dx \, dy. \qquad (6)$$

The region Ω can be used to restrict the functional to only part of the surface in question, thereby localizing the smoothing effect. Figure 1 shows a truncated cone which has been capped in such a way as to minimize the thin plate energy of the capping surface and form a C^1 join with the truncated cone.

When actually minimizing this functional, we form a linear system similar to the normal equations used for least squares minimization. If we define $N_I = N_{i,j,k}^{\Delta}$ and $N_J = N_{l,m,n}^{\Delta'}$ to be two DMS basis functions (not necessarily distinct) then the individual entries $E_{I,J}$ of this matrix are of the form:

$$\int_\Omega N_{I_{xx}} N_{J_{xx}} + 2N_{I_{xy}} N_{J_{xy}} + N_{I_{yy}} N_{J_{yy}} \, dx \, dy. \qquad (7)$$

In order to find this matrix, we must evaluate this integral. We can use Equation 3 to express the second derivative of each DMS basis function in terms of simplex splines of lower order. Once expressions for the second derivatives of two basis functions have been obtained, we can find the integral of their product. It is conceivable to do this numerically, but we would like to represent these second derivatives symbolically, so that the integral can be found exactly.

If the DMS spline is of degree two, then each of its second derivatives is piecewise constant. It follows from Equation 3 that each second derivative can be expressed as the sum of many special piecewise constant functions, where each special function is a non-zero constant within a (half-open) triangular region of the domain and vanishes outside it. Let $S_I = \sum_i T_i$ and $S_J = \sum_j T_j$ be second derivatives of N_I and N_J, respectively, represented as sums of special piecewise constant functions. The integral of their product can be rewritten as:

$$\begin{aligned} \int_\Omega S_I S_J \, dx \, dy &= \int_\Omega (\sum_i T_i)(\sum_j T_j) \, dx \, dy \\ &= \sum_i \sum_j \int_\Omega T_i T_j \, dx \, dy \qquad (8) \end{aligned}$$

Integration of the product of two special piecewise constant functions can be performed by clipping the domain triangle of one against the domain triangle of the other. The resulting polygon is then clipped against the Ω region and the area of the result is multiplied by the heights of both piecewise constants (in [Tra90], the intersection of two triangles is found using linear programming). The partial results are then summed to form the final integral. Since the regions of support of most DMS basis functions will be disjoint from one another, we can accelerate the calculation if we only compute the product of special functions that lie in the intersection of the bounding boxes of the two basis functions N_I and N_J. This method deals with the integration problem uniformly for degree two surfaces, no matter how the knots of any particular basis function are configured.

The smoothing functional cannot be used on its own for scattered data fitting. However, it is easy to combine thin plate energy with least squares and obtain a functional that, when minimized, tries to smooth the surface while fitting the surface to the scattered data.

4.3 Combining least squares and smoothing

One way of combining a smoothing factor with our least squares functional is to form a linear combination of the original least squares functional and the thin plate energy functional [BHS93]:

$$LSJ(F) = (1-\alpha)\, LS(F) + \alpha\, J(F), \quad 0 \leq \alpha \leq 1 \quad (9)$$

By changing the value of α, we can vary the relative strengths of the least squares approximation and the thin plate energy terms. The least squares term ensures that the surface approximates the given data points, while the smoothness term ensures that the surface maintain a certain degree of smoothness, and that that basis functions that are either underdetermined or linearly dependent with respect to the data are assigned values minimizing bending energy (see Figures 2 and 3)

The process of finding an approximating surface can now be summarized. First, the data points are used to define a triangulation, based on the quadtree approach outlined above. Knot clouds are then assigned to each vertex of the triangulation. This defines the DMS basis functions, which gives us a space of functions in which to find our approximating surface F.

In both the purely least squares and mixed least squares thin plate energy cases, we evaluate the set of basis functions at each sample (x, y) location. When minimizing a purely least squares problem, these values are assembled into an observation matrix and the basis function coefficients are calculated using linear algebra techniques.

If we wish to solve a least squares with smoothing problem, then the second derivatives of the basis functions must also be evaluated symbolically as outlined in Section 4.2 using the bounding box of the quadtree as the Ω region. The results of integration are then placed into a matrix, combined with the normal equations assembled for the purely least squares solution (taking the α factor into account), and the basis function coefficients are found using linear algebra techniques.

5 Conclusions

We have presented a way of using DMS spline surfaces for functional scattered data approximation. The method generates a triangulation of the parameter domain which, in turn, defines a set of basis functions adapted to the local density of the data. A surface is then found using either a pure least squares method or a combination of least squares and thin-plate energy minimization. The combined method allows data sets to be fit which cannot be fit using the least squares method alone.

Further work should be done in the following areas. By placing certain knots collinearly it is possible to have some of a DMS spline surface's triangular patches meet with lower-than-maximal continuity. This is useful for modeling known discontinuities in the data. This scheme should be extended to incorporate into the triangulation parametric edges where lower continuity is desired.

The possibility of finding a different triangulation method that completely addresses our triangulation goals should be more fully examined. The question of finding a "fair" triangulation (in the sense of Property Two) should be more completely explored, as it is likely that it could be used in a wider setting to characterize attributes of a data set.

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